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Version of Record: https://www.sciencedirect.com/science/article/pii/S0951832021002325
Manuscript_95cfb0a3355cc37441b31b9005da12d4

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Abstract

Predictive maintenance is a promising solution to keep the long-run operation of industrial systems at high reliability and low cost. In this spirit, we aim to develop an adaptive predictive maintenance model for continuously deteriorating single-unit systems subject to periodic inspection, imperfect repair and perfect replacement. The development consists of four steps: degradation modeling, maintenance effect modeling, maintenance policy elaboration, and performance evaluation. Compared with existing models, ours differs in three main aspects. Firstly, we take into account the past dependency of maintenance actions in the degradation modeling via the random effect of an inverse Gaussian process. Secondly, we use both the system remaining useful life and maintenance duration to enable dynamic maintenance decision-making. Finally, we take advantage of the semi-regenerative theory to analytically evaluate the long-run cost rate of maintenance policies whose decision variables are of different nature. We validate and illustrate the developed adaptive predictive maintenance model by various numerical experiments. Comparative studies with benchmarks under different maintenance costs and degradation characteristics confirm the flexibility and cost-effectiveness of the model.

Keywords: Adaptive maintenance decision; Imperfect repair; Inverse Gaussian process; Predictive maintenance; Remaining useful life; Semi-regenerative process

Acronyms

AGAN	as-good-as-new
cdf, pdf, sf	cumulative distribution function, probability density function, survival function
CR, PR	corrective replacement, preventive replacement
IG	inverse Gaussian
IM, IR	imperfect maintenance, imperfect repair
PdM	predictive maintenance
RUL	remaining useful life
std	standard deviation

Notations

X_t, $\{X_t\}_{t\geq 0}$	system deterioration level at time $t \geq 0$, system degradation process
$\mathcal{IGP}(\cdot,\cdot)$, $\mathcal{IG}(\cdot,\cdot)$,	IG process, IG distribution
E_j, S_j	end time of the j-th repair/replacement, starting time of the $(j+1)$-st repair/replacement
μ_0, $\mu_1\left(x_j\right)$	constant and x_j-dependent parts of the shape parameter $\mu\left(x_j\right)$ of IG process $\{X_t\}_{E_j^+\leq t\leq S_j}$ with $X_{E_j^+}=x_j$
λ	constant scale parameter of $\{X_t\}_{t\geq 0}$
$\Phi(\cdot)$, $\varphi(\cdot)$,	standard normal cdf and pdf
$T_{j,k}$	k-th inspection time during the $(j+1)$-st repair/replacement cycle $\left[E_j^+, E_{j+1}^+\right]$
$f(\cdot;\cdot,\cdot)$, $F(\cdot;\cdot,\cdot)$,$\bar{F}(\cdot;\cdot,\cdot)$	pdf, cdf and sf of IG probability law
L, $R_{T_{j,k}}$	system failure threshold, system RUL at $T_{j,k}$
$\bar{F}_{R_{T_{j,k}}\mid\cdot,\cdot}$, $S\left(R_{T_{j,k}}\mid\cdot,\cdot\right)$	sf and std of $R_{T_{j,k}}$
ρ_0, $\rho_1(\cdot,\cdot)$	constant duration for a PR or CR, additional degradation-dependent duration for an IR
$\rho(\cdot,\cdot)$	total duration for an IR
$g(\cdot\mid\cdot,\cdot)$	truncated pdf of the system deterioration level after an IR
C_m, C_r, C_o, C_i, C_u	inspection cost, repair cost, replacement cost, inactivity cost rate and unavailability cost rate

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April 13, 2021

$C(t)$, C_∞	cumulative maintenance cost up to time t, long-run maintenance cost rate
$N_m(t)$, $N_m\left(\left[E_j^+, E_{j+1}^+\right]\right)$	number of inspections up to time t, and over $\left[E_j^+, E_{j+1}^+\right]$
$N_r(t)$, $N_r\left(\left[E_j^+, E_{j+1}^+\right]\right)$	number of repairs up to time t, and over $\left[E_j^+, E_{j+1}^+\right]$
$N_o(t)$, $N_o\left(\left[E_j^+, E_{j+1}^+\right]\right)$	number of replacements up to time t, and over $\left[E_j^+, E_{j+1}^+\right]$
$I(t)$, $I\left(\left[E_j^+, E_{j+1}^+\right]\right)$	cumulative duration of the system inactivity up to time t, and over $\left[E_j^+, E_{j+1}^+\right]$
$U(t)$, $U\left(\left[E_j^+, E_{j+1}^+\right]\right)$	cumulative duration of the system unavailability up to time t, and over $\left[E_j^+, E_{j+1}^+\right]$
ΔE_{j+1}	length of the $(k+1)$-st Markov renewal cycle $\left[E_j^+, E_{j+1}^+\right]$
$\psi(\cdot,\cdot,\cdot)$	waiting time interval before a maintenance
δ, σ, α, τ	decision variables of the $(\delta, \sigma, \alpha, \tau)$ policy
δ, ξ, ω, η	decision variables of the $(\delta, \xi, \omega, \eta)$ policy
$\left\{Y_j\right\}_{j\geq\mathbb{N}}$	Markov chain describing the system deterioration at repair/replacement times ($Y_j = X_{E_j^+}$)
$\pi(\cdot)$	stationary measure of $\left\{Y_j\right\}_{j\geq\mathbb{N}}$
$P(\cdot,\cdot)$	transition kernel of $\left\{Y_j\right\}_{j\geq\mathbb{N}}$
$E_\pi[\cdot]$	expectation with respect to the stationary measure π
$1_{\{\cdot\}}$, $\delta_0(\cdot)$	indicator function, Dirac mass at 0

1. Introduction

With usage, age and environmental impacts, many industrial systems suffer continuous degradation leading eventually to random failure, no matter how good they are designed. Cutting tools are subject to cumulative wear [1], hydrocarbon pipelines undergo corrosion [2], hydraulic structures suffer erosion [3], water-feeding turbo-pumps of nuclear power plants endure fatigue crack growth [4], etc. The failure of such systems causes damage not only to the industry, but also to the society and the environment. Therefore, keeping their efficient operation at high reliability is vital to enterprises, especially in the context of aggressive global competition. When the systems are repairable, various maintenance actions, such as inspection, testing, repair, replacement, etc., are conceivable [5]. Nevertheless, these actions are themselves time-consuming and costly, seeking an effective maintenance solution to gain a competitive advantage is thus of great concern to enterprises.

Among existing solutions (see e.g., [6] for a recent overview), predictive maintenance (PdM) could be most appropriate [7]. It uses collected condition monitoring data to predict the future system health in real-time, and through that knowledge, enables maintenance decision-making [8]. As such, the PdM allows to save resources by carrying out proper and timely actions only when necessary. This definite advantage, together with the dissemination of computer-based monitoring technologies, has promoted the rapid development of PdM models for continuously deteriorating systems during the last two decades [9]. Basically, the PdM modeling includes four connected steps [10]: (*i*) continuous degradation modeling, (*ii*) maintenance effects modeling, (*iii*) maintenance policy elaboration, and (*iv*) performance assessment.

1.1. State-of-the-art

Continuous degradation modeling is critical for system health prognostics. In the current state-of-the-art, it relies mostly on Lévy or diffusion stochastic processes that fit in with condition monitoring data [11]. The Gamma process is the most common choice for monotonic degradation because of its physical meaning and mathematical tractability [12]. When this choice fails (e.g., in the case of GaAs laser degradation data [13] or energy pipeline corrosion data [14]), the inverse Gaussian (IG) process [15] could be a good alternative. Indeed, it is proved more flexible than the Gamma process in incorporating random effects and covariates, while still holding similar physical and mathematical meanings [16]. This explains why the IG process has recently attracted considerable attentions, especially in accelerated degradation test optimization [17] and in the system remaining useful life (RUL) assessment [18]. However, up to now, its applications in PdM modeling are still very scarce despite its versatility [19].

Maintenance effects modeling describes the impacts of maintenance actions on the system degradation behavior. Perfect maintenance with as-good-as-new (AGAN) effect is certainly the best-known assumption in the literature [20]. However, it cannot cover numerous realistic actions whose imperfectness may be induced by influential factors such as human errors, spare parts quality, lack of materials, lack of maintenance time, etc. To meet this practical need, imperfect maintenance (IM) models (see [21] for a complete overview) have been extensively studied under the assumption that the system condition after a maintenance is worse-than-new but better-than-old. Despite those efforts, modeling IM effects in the context of degradation processes is still not active, and mainly mimics the ideas of lifetime-based IM models (see e.g., [22–25]). Furthermore, in most existing models, past and present IM actions are linked by the memory assumption [26]. This assumption is however not easily verified in practice owing to the stochastic nature of degradation processes. How to break this strong assumption to reach more feasible past-dependent IM models remains an open issue.

PdM policy specifies actions to be adopted at decision epochs based on the system health evolution. In the literature, the elaboration of such a policy has mainly focused on inspection schedules and thresholds for preventive maintenance decision-making [27]. A policy with periodic inspection and fixed degradation-based threshold is perhaps the simplest one (see e.g., [28, 29]). This simplification allows easy applications in practice, but also implies a non-optimal maintenance policy. Therefore, much attentions have been paid to make the policy more flexible seeking higher performance. Some authors suggested non-periodic inspection schemes by adapting the inter-inspection intervals to the system age [30], system degradation level [31], system degradation rate [32], system RUL [33], or to the working environment [34]. Meanwhile, some others proposed using varied preventive maintenance thresholds (see e.g., [19, 35]). These solutions usually lead to very complex, even unfeasible, maintenance policies because of a high number of decision variables. So, a big challenge is to build an adaptive PdM policy, on the one hand, simple enough to facilitate practical applications, and on the other hand, sophisticated enough to fulfill certain performance criteria (e.g., cost, availability, etc.).

The ultimate goal of performance assessment is to optimize maintenance policies by adjusting their decision variables (inter-inspection intervals, maintenance thresholds, etc.) so that the maintenance performance is maximum. As far as economic criteria and mathematical models are concerned, the performance assessment can be done by two main approaches [36]. The first approach relies on the (semi)-Markov decision process and dynamic programming or policy iteration tools [37]. It requires transforming the continuous-state space of degradation processes into the associated discrete-state space [19, 38]. Note, however, that such a discretization could be undesirable in some cases where the intrinsic continuity of degradation processes is more significant for maintenance decision-makers [10]. The second approach allows avoiding this obstacle. Indeed, using the (semi)-regenerative theory or (Markov)-renewal theory [39], this approach can be effectively applied for degradation processes with continuous-state space [40], as well as with discrete-state space [41]. More importantly, it results in full analytical cost models rather than numerical solutions given by the first approach. Recently, several authors have used the semi-regenerative technique to evaluate and optimize the long-run maintenance cost rate of static maintenance policies (see e.g., [42, 43]). This technique is even more meaningful in the context of adaptive PdM policies [40, 44], but more effort in mathematical formulation is, no doubt, required [45].

1.2. Contributions and organization

In this paper, our ambition is to seek answers to the above-mentioned shortcomings of the PdM modeling. Consequently, a new adaptive PdM model is developed for a continuously deteriorating single-unit system subject to periodic inspection, imperfect repair (IR) and perfect replacement. The inspection and replacement are assumed memoryless, while the repair is past-dependent in the sense that it cannot bring the system back to a degradation level better than the one reached at the last repair [29]. To characterize such an imperfect effect, we sample the system degradation level after a repair from a probability distribution truncated by the degradation levels just before the current repair and just after the last repair/replacement. This allows breaking the memory assumption usually made in the IM modeling (see e.g., arithmetic reduction model [22] or Kijima's type model [25]). We also take into account the impact of maintenance efficiency on the system degradation behavior via the random effect of an IG degradation process. Indeed, expressing its shape parameter as an increasing function of the degradation level returned by a system repair/replacement, we consider that the system degradation is faster and more chaotic when the maintenance efficiency is lower. Some real-world examples of this phenomenon are the degradation paths of gyroscopes and draught fans provided respectively in [46] and [47]. Besides, the considered maintenance actions consume different duration and incur different costs. All of them should be properly coordinated into a unifying policy to enable the best maintenance performance. Using the system RUL and the maintenance duration as decision variables, we build an adaptive PdM policy that allows (*i*) to determine the right moment for a switch from periodic inspection to a maintenance, (*ii*) to dynamically schedule maintenance times, and (*iii*) to select the suitable maintenance action (i.e., either IR or perfect replacement) at a scheduled time. Interestingly, the versatility of decision variables offers a good compromise between simplicity and performance. To assess the economic performance of the policy in the long-run, we analyze the probabilistic behavior of the maintained system at steady state, and analytically evaluate its maintenance cost rate using the semi-regenerative technique. The main difficulty lies in the difference in the nature of decision variables. We thus transform all these variables into associated degradation levels to overcome this obstacle. Various numerical experiments and comparative studies with benchmarks under different system configurations allow to illustrate the developed adaptive PdM model, and to confirm its cost-effectiveness.

The paper remainder is organized as follows. Section 2 describes the considered maintained system. Its probabilistic behavior at steady state is analyzed in Section 3. Sections 4 is devoted to the cost model formulation and optimization for the maintained system. Section 5 gives more insight into the effectiveness of the proposed adaptive PdM model thanks to numerical assessments. Finally, conclusions and perspectives are discussed in Section 6.

2. Description of the maintained system

Let us consider a continuously deteriorating single-unit system subject to inspection, imperfect repair and perfect replacement. Its health state at time $t \geq 0$ is completely characterized by a scalar random variable $X_t \geq 0$, with $x_j = 0$. In the absence of repair or replacement, the system evolves from its new state (i.e., $X_t = 0$) to its failure state (i.e., $X_t \geq L$) following an increasing stochastic degradation process $\{X_t\}_{t\geq 0}$. Each of maintenance actions has its own effects on the behavior of $\{X_t\}_{t\geq 0}$ and incurs different cost. From an economic viewpoint, they should be coordinated into a unified adaptive PdM policy. The aim of this section is to model such a maintained system based on the assumption that $\{X_t\}_{t\geq 0}$ acts like an inverse Gaussian process. To facilitate the comprehension, the following description focuses on the $(j+1)$-st repair/replacement cycle which is the time interval between two successive end-of-repair/replacement times E_j and E_{j+1}, with $E_0 = 0$ and $j \in \mathbb{N}$.

2.1. Degradation modeling

Let S_j be the starting time of a repair/replacement within the cycle $\left[E_j^+, E_{j+1}^+\right]$, the system degradation over $\left[E_j^+, S_j\right]$ evolves from $X_{E_j^+} = x_j$ following a stationary IG process with shape parameter $\mu\left(x_j\right)$ and scale parameter $\lambda > 0$

$$\{X_t\}_{E_j^+ \leq t < S_j} \sim \mathcal{IGP}\left(\mu\left(x_j\right), \lambda\right). \tag{1}$$

The shape function $\mu\left(x_j\right)$ is the sum of two elements: (*i*) a constant $\mu_0 > 0$ characterizing the proper dynamics of the system degradation, and (*ii*) a function $\mu_1\left(x_j\right) \geq 0$ representing the effects of repair/replacement actions before E_j on the dynamics of $\{X_t\}_{E_j^+ \leq t < S_j}$ via the recovery degradation level x_j

$$\mu\left(x_j\right) = \mu_0 + \mu_1\left(x_j\right). \tag{2}$$

As in [48], we shall express $\mu_1\left(x_j\right)$ as an increasing function of x_j with $\mu_1(0) = 0$ to take into account the phenomenon that the lower the efficiency of past repairs, the more the system is chaotic and vulnerable to degradation. Therefore, the degradation increment between two times s and t, with $E_j^+ \leq s < t \leq S_j$, is IG distributed with shape parameter $\mu\left(x_j\right)\cdot(t-s)$ and scale parameter $\lambda \cdot (t-s)^2$, i.e., $X_t - X_s \sim \mathcal{IG}\left(\mu\left(x_j\right)\cdot(t-s), \lambda\cdot(t-s)^2\right)$. Its probability density function (pdf), cumulative distribution function (cdf) and survival function (sf) are respectively

$$f\left(x; \mu\left(x_j\right)\cdot(t-s), \lambda\cdot(t-s)^2\right) = \sqrt{\frac{\lambda\cdot(t-s)^2}{2\pi x^3}} \cdot \exp\left(-\frac{\lambda}{2\mu^2\left(x_j\right)} \cdot \frac{\left(x - \mu\left(x_j\right)\cdot(t-s)\right)^2}{x}\right) \cdot 1_{\{x>0\}}, \tag{3}$$

$$F\left(x; \mu\left(x_j\right)\cdot(t-s), \lambda\cdot(t-s)^2\right) = \Phi\left(\sqrt{\frac{\lambda}{x}} \cdot \left(\frac{x}{\mu\left(x_j\right)} - (t-s)\right)\right) + \exp\left(\frac{2\lambda\cdot(t-s)}{\mu\left(x_j\right)}\right) \cdot \Phi\left(-\sqrt{\frac{\lambda}{x}} \cdot \left(\frac{x}{\mu\left(x_j\right)} + (t-s)\right)\right), \tag{4}$$

$$\bar{F}\left(x; \mu\left(x_j\right)\cdot(t-s), \lambda\cdot(t-s)^2\right) = \Phi\left(-\sqrt{\frac{\lambda}{x}} \cdot \left(\frac{x}{\mu\left(x_j\right)} - (t-s)\right)\right) - \exp\left(\frac{2\lambda\cdot(t-s)}{\mu\left(x_j\right)}\right) \cdot \Phi\left(-\sqrt{\frac{\lambda}{x}} \cdot \left(\frac{x}{\mu\left(x_j\right)} + (t-s)\right)\right), \tag{5}$$

where $1_{\{\cdot\}}$ is the indicator function which equals 1 if the argument is true and 0 otherwise, $\Phi(\cdot)$ denotes the standard normal cdf

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(-\frac{t^2}{2}\right) \mathrm{d}t. \tag{6}$$

These probability functions are the basis to develop the stationary law and the cost model of the maintained system (see Sections 3 and 4). The IG degradation process $\mathcal{IGP}\left(\mu\left(x_j\right), \lambda\right)$ has the mean rate $\mu\left(x_j\right)$ and the variance rate $\mu^3\left(x_j\right)/\lambda$ over $\left[E_j^+, E_{j+1}^+\right]$. Despite its stationarity over a given repair/replacement cycle, this process is non-stationary from cycle to cycle owing to the randomness of x_j, $j \in \mathbb{N}$.

2.2. Inspection and remaining useful life estimation

The inspection is merely an information-taking operation that reveals perfectly the current hidden degradation level of the system. We assume that the inspection takes negligible time and has no effect on the system degradation behavior. This is why the system degradation level just before and just after a k-th inspection time, $k \in \mathbb{N}^*$, during the cycle $\left[E_j^+, E_{j+1}^+\right]$ remains unchanged (i.e., $X_{T_{j,k}^-} = X_{T_{j,k}^+} = X_{T_{j,k}}$).

Given $X_{T_{j,k}}$, we can further access the system RUL using the degradation model $\mathcal{IGP}\left(\mu\left(x_j\right), \lambda\right)$. Indeed, let L denote a fixed failure threshold, if no more maintenance action is planned, the system RUL at $T_{j,k}$ given $X_{T_{j,k}} = x_{j,k} < L$ and $X_{E_j^+} = x_j < x_{j,k}$ is defined by the random variable

$$R_{T_{j,k}} = \inf\left\{r > 0 : X_{T_{j,k}+r} \geq L \mid X_{T_{j,k}} = x_{j,k}, X_{E_j^+} = x_j\right\}. \tag{7}$$

We are interested in the sf and the standard deviation (std) of $R_{T_{j,k}}$ which are measures of the *predictive system reliability* and of the *prognosis precision* respectively. The increasing of $\mathcal{IGP}\left(\mu\left(x_j\right), \lambda\right)$ allows to compute the sf of $R_{T_{j,k}}$ at r as

$$\bar{F}_{R_{T_{j,k}}|X_{T_{j,k}},X_{E_j^+}}\left(r \mid x_{j,k}, x_j\right) = P\left(X_{T_{j,k}+r} < L \mid X_{T_{j,k}} = x_{j,k}, X_{E_j^+} = x_j\right) = F\left(L - x_{j,k}; \mu\left(x_j\right) \cdot r, \lambda \cdot r^2\right), \tag{8}$$

where $F_{X_{s+r}-X_s}(\cdot)$ is given by (4). Meanwhile, the std of $R_{T_{j,k}}$ can be obtained from $\bar{F}_{R_{T_{j,k}}|X_{T_{j,k}},X_{E_j^+}}\left(r \mid x_{j,k}, x_j\right)$ as follows

$$S\left(R_{T_{j,k}} \mid X_{T_{j,k}} = x_{j,k}, X_{E_j^+} = x_j\right) = \frac{\mu\left(x_j\right)}{\lambda}\sqrt{\frac{1}{4} + \beta^2\left(\beta^2+3\right)\Phi(\beta) + \beta\left(\beta^2+2\right)\varphi(\beta) - \left(\left(\beta^2+1\right)\Phi(\beta) + \beta\varphi(\beta)\right)^2}, \tag{9}$$

where $\beta := \beta\left(x_{j,k}, x_j\right) = \sqrt{\lambda \cdot \left(L - x_{j,k}\right)}/\mu\left(x_j\right)$. We remark that both $\bar{F}_{R_{T_{j,k}}|X_{T_{j,k}},X_{E_j^+}}\left(r \mid x_{j,k}, x_j\right)$ and $S\left(R_{T_{j,k}} \mid X_{T_{j,k}} = x_{j,k}, X_{E_j^+} = x_j\right)$ are independent of the inspection time $T_{j,k}$, and always decreasing with respect to $x_{j,k}$ or x_j (see also Subsection 2.6). This property makes these measures more versatile for predictive maintenance than the degradation levels $X_{T_{j,k}}$ and $X_{E_j^+}$. We shall use $S\left(R_{T_{j,k}} \mid X_{T_{j,k}} = x_{j,k}, X_{E_j^+} = x_j\right)$ to decide a switching from an inspection to a maintenance at an inspection time $T_{j,k}$, and $\bar{F}_{R_{T_{j,k}}|X_{T_{j,k}},X_{E_j^+}}\left(r \mid x_{j,k}, x_j\right)$ for maintenance planning because of the time mark r.

2.3. *Replacement and repair*

The replacement is a perfect action that always makes the system AGAN independently of previous system states and interventions. It takes a fixed duration ρ_0 due to, e.g., the material set-up, the system dismantling and reassembly, etc. During the replacement duration, the system is assumed inactivated. Therefore, if a replacement starts at S_j, it will end at $E_{j+1} = S_j + \rho_0$. The corresponding system degradation levels are $X_{S_j} = X_{E_{j+1}^-}$ and $X_{E_{j+1}^+} = 0$.

The repair, on the contrary, is an IM action whose duration and efficiency depend closely on the system degradation levels returned by the last replacement/repair $X_{E_j^+}$ and at the beginning of the current repair X_{S_j}. More concretely, the repair duration includes not only a fixed duration ρ_0 due to the same reasons as in the replacement, but also a variable duration $\rho_1\left(X_{E_j^+}, X_{S_j}\right)$ depending on X_{S_j} and $X_{E_j^+}$

$$\rho\left(X_{E_j^+}, X_{S_j}\right) = \rho_0 + \rho_1\left(X_{E_j^+}, X_{S_j}\right), \tag{10}$$

with $\rho_1(0,0) = 0$. Especially, $\rho_1\left(X_{E_j^+}, X_{S_j}\right) \geq 0$ is an increasing function of $X_{E_j^+}$ and $X_{S_j^+}$, because a higher degraded system (i.e., higher X_{S_j}) which has been undergone more repairs since the last renewal (hence higher $X_{E_j^+}$) requires naturally a longer repair duration [44]. Throughout the repair, the system is inactivated: $X_{S_j} = X_{E_{j+1}^-}$ with $E_{j+1} = S_j + \rho\left(X_{E_j^+}, X_{S_j}\right)$. Once the repair ends, the system degradation is returned to a random level between $X_{E_j^+}$ and X_{S_j} (i.e., $X_{E_j^+} \leq X_{E_{j+1}^+} \leq X_{S_j}$). Such a condition implies the impact of previous maintenance actions on the current repair (i.e., $X_{E_j^+} \leq X_{E_{j+1}^+}$) while retaining the inherent better-than-old characteristic (i.e., $X_{E_{j+1}^+} \leq X_{S_j^+}$) [21]. Evidently, a system, which has been repaired several times since the last renewal, is subject to lower efficiency even the same repair quality [29, 49, 50]. To model such a repair efficiency, we represent $X_{E_{j+1}^+}$ as a random sample of a truncated pdf with lower bound $X_{E_j^+}$ and upper bound $X_{E_{j+1}^-}$ [51]

$$X_{E_{j+1}^+} \sim g\left(x_{j+1} \mid X_{E_j^+}, X_{S_j}\right). \tag{11}$$

This model allows avoiding the strong memory assumption usually made in the IM modeling: *the system after a repair is put back to an exact deterioration level where it was in the past* (see e.g., [22] and [25]).

2.4. *Maintenance costs*

We denote C_m, C_r and C_o the constant unit costs for inspection, repair and replacement respectively. C_r and C_o include already the inspection cost. The effects of maintenance actions on the system degradation imply the relationship $C_o > C_r > C_m > 0$. Besides, depending on the system state at a maintenance time, additional costs could be considered. During a preventive maintenance, a running system is inactivated at a constant cost rate C_i. Whereas, a failed system induces an unavailability cost at constant rate C_u from the failure time to the end of the next corrective maintenance (i.e., also including the required duration for the corrective maintenance). Since the system unavailability is unforeseen, it incurs a higher cost rate than the system inactivity (i.e., $C_u > C_i > 0$). Therefore, the cumulative maintenance cost incurred in the time interval $[0, t]$ can be expressed as

$$C(t) = C_m \cdot N_m(t) + C_r \cdot N_r(t) + C_o \cdot N_o(t) + C_i \cdot I(t) + C_u \cdot U(t), \tag{12}$$

where $N_m(t)$, $N_r(t)$ and $N_o(t)$ denote respectively the number of inspections, of repairs and of replacements up to time t, $I(t)$ and $U(t)$ are the total duration of system inactivity and unavailability in $[0, t]$. Our aim is to build an adaptive PdM policy allowing the lowest long-run maintenance cost rate [52, chapter 11]

$$C_\infty = \lim_{t\to\infty} \frac{C(t)}{t}, \tag{13}$$

with $C(t)$ given from (12). Note that C_m, C_r and C_o are given in *cost unit*, C_i, C_u and C_∞ are given in *cost unit/time unit*, and $I(t)$, $U(t)$ are given in *time unit*. The cost unit may be U.S. Dollar, Euro, British Pound, etc., while time unit may be month, quarter, year, etc. The cost unit may be U.S. Dollar, Euro, British Pound, etc., and time unit may be month, quarter, year, etc.

2.5. *Policy structure*

Seeking the lowest C_∞ leads to two questions: (*i*) when should we intervene on the system? and (*ii*) what should we do at an intervention time? Following the PdM concept, we rely on the std of RUL to decide, at a periodic inspection time, whether to continue inspection or to switch to maintenance. If the latter is chosen, we next use the sf of RUL to determine a proper maintenance time. The nature of maintenance operation (i.e., repair or replacement) will depend on associated maintenance duration. As in [53], we prefer using a parametric structure (with decision variables δ, σ, α and τ) to build the desired PdM policy. Consequently, the maintenance plans and decision rules over the $(j+1)$-st maintenance cycle $\left[E_j^+, E_{j+1}^+\right]$, with $X_{E_j^+}$, are schematized in Figure 1.

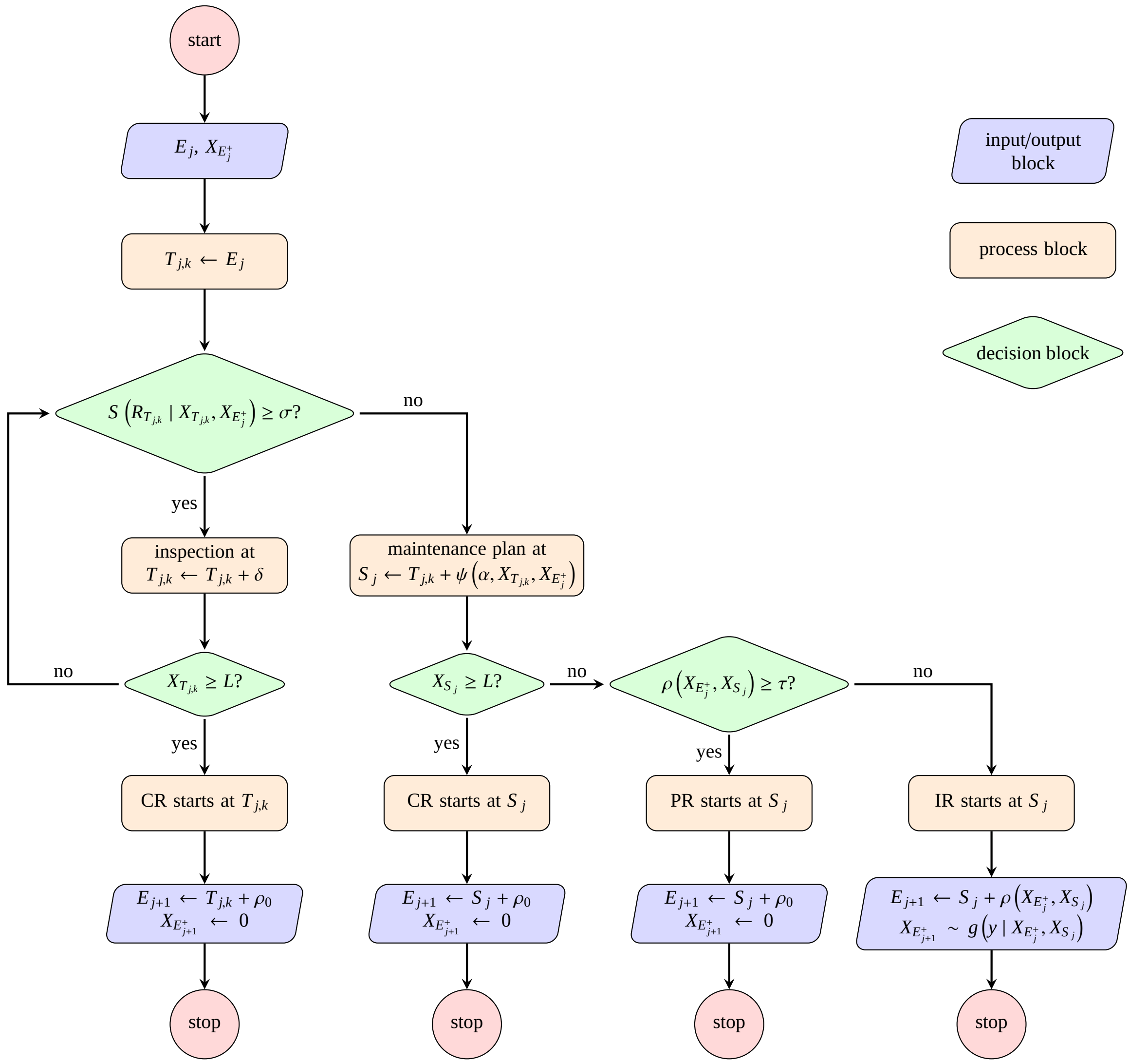


Figure 1: Maintenance plan and decision rules over the $(j+1)$-st maintenance cycle $\left[E_j^+, E_{j+1}^+\right]$

The system is regularly inspected at times $T_{j,k} = E_j + k \cdot \delta$, with $k = 0, 1, 2, \ldots$, and inspection period $\delta > 0$. Note that the moment $T_{j,0} = E_j$ of the cycle $\left[E_j^+, E_{j+1}^+\right]$ is just a "fictional" inspection time, at which the system degradation level has already known by the last repair/replacement $X_{T_{j,0}^+} = X_{E_j^+}$. This is why no cost is counted at $T_{j,0} = E_j$. Given $X_{T_{j,k}}$, the following rules are adopted.

1. If $X_{T_{j,k}} \geq L$, the system has been failed within the interval $\left(T_{j,k-1}, T_{j,k}\right]$, and a corrective replacement (CR) starts immediately at $T_{j,k}$. After the CR, the system is AGAN (i.e., $X_{E_{j+1}^+} = 0$ with $E_{j+1} = T_{j,k} + \rho_0$).

2. If $X_{T_{j,k}} < L$ and $S\left(R_{T_{j,k}} \mid X_{T_{j,k}}, X_{E_j^+}\right) < \sigma$, the system is still running at $T_{j,k}$ and its RUL can be predicted with an acceptable precision. No further inspection is thus needed, and the next maintenance is planned at $S_j = T_{j,k} + \psi\left(\alpha, X_{T_{j,k}}, X_{E_j^+}\right)$ with

$$\psi\left(\alpha, X_{T_{j,k}}, X_{E_j^+}\right) = \left\{r \geq 0 \mid \bar{F}_{R_{T_{j,k}}|X_{T_{j,k}},X_{E_j^+}}\left(r \mid x_{j,k}, x_j\right) = \alpha\right\} \tag{14}$$

and $0 \leq \alpha \leq 1$. The decrease of $\bar{F}_{R_{T_{j,k}}|X_{T_{j,k}},X_{E_j^+}}\left(r \mid x_{j,k}, x_j\right)$ allows dynamic maintenance times adaptive to the degradation levels $X_{E_j^+}$ and $X_{T_{j,k}}$ while keeping the system reliability at a desired level α. The choice of a maintenance action depends on the degradation level X_{S_j} and on the associated maintenance duration.

   (a) If $X_{S_j} \geq L$, the system has been failed within the interval $\left(T_{j,k}, S_j\right]$, a CR with duration ρ_0 is triggered immediately at S_j to reset the system to an AGAN state (i.e., $X_{E_{j+1}^+} = 0$ with $E_{j+1} = S_j + \rho_0$).

   (b) If $X_{S_j} < L$ and $\rho\left(X_{E_j^+}, X_{S_j}\right) \geq \tau$, the system is still running at S_j, and the "long" duration $\rho\left(X_{E_j^+}, X_{S_j}\right)$ implies that an imperfect repair (IR) is not appropriate for the current maintenance. This is why a preventive replacement (PR) should be carried out immediately at S_j. After the PR, the system is AGAN (i.e., $X_{E_{j+1}^+} = 0$ with $E_{j+1} = S_j + \rho_0$).

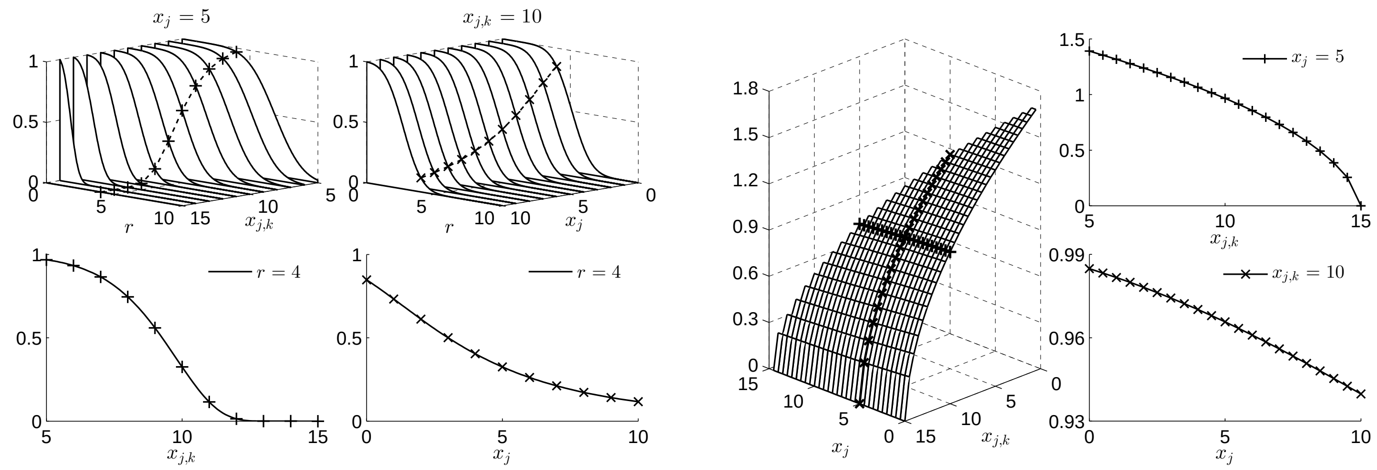
   (c) If $X_{S_j} < L$ and $\rho\left(X_{E_j^+}, X_{S_j}\right) < \tau$, we perform an IR immediately at S_j on the running system because of its economic efficiency. After the preventive IR, the system degradation $X_{E_{j+1}^+}$ returns to a random level between $X_{E_j^+}$ and X_{S_j} such that $X_{E_{j+1}^+} \sim g\left(y \mid X_{E_j^+}, X_{S_j}\right)$, where $E_{j+1} = S_j + \rho\left(X_{E_j^+}, X_{S_j}\right)$ and g is a known truncated pdf.

3. If $X_{T_{j,k}} < L$ and $S\left(R_{T_{j,k}} \mid X_{T_{j,k}}, X_{E_j^+}\right) \geq \sigma$, additional inspections are needed to reinforce the precision of RUL prediction. Accordingly, the decisions is postponed to the next inspection time at $T_{j,k+1} = T_{j,k} + \delta$.

The next maintenance cycle begins at E_{j+1}^+ with initial deterioration level $X_{E_{j+1}^+}$. For this maintenance policy, the decision variables δ, σ, α and τ are parameters to be optimized. To highlight their importance, we call the policy $(\delta, \sigma, \alpha, \tau)$.

2.6. Numerical illustration

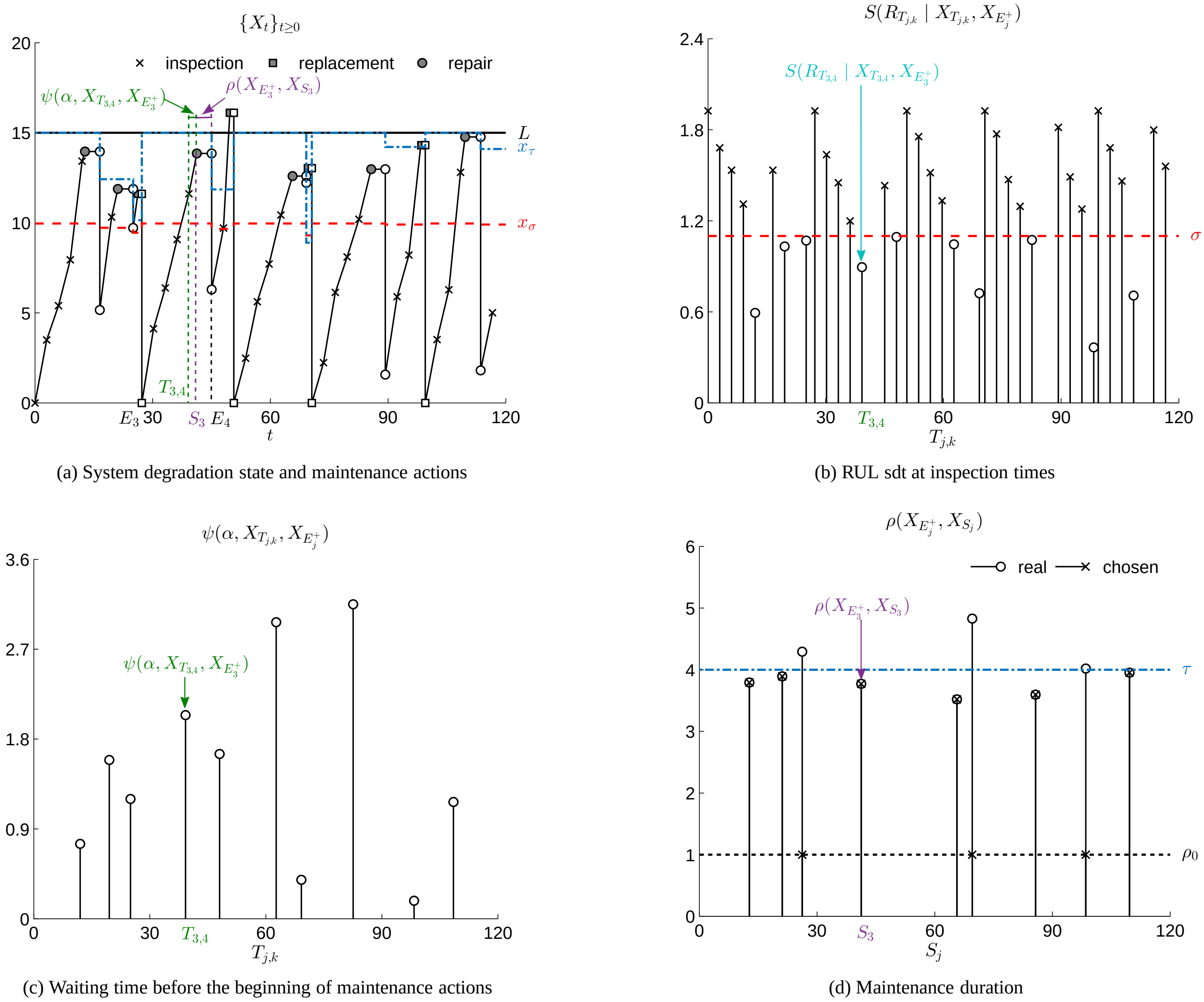
For a numerical illustration, we consider a maintained single-unit system defined by $L = 15$, $\mu\left(X_{E_j^+}\right) = 1 + 0.1 \cdot X_{E_j^+}$, $\lambda = 4$, $\rho\left(X_{E_j^+}, X_{S_j}\right) = \rho_0 + \rho_1\left(X_{E_j^+}, X_{S_j}\right) = 1 + 0.1 \cdot X_{E_j^+} + 0.2 \cdot X_{S_j}$, a continuous uniform pdf for $g\left(x_{j+1} \mid X_{E_j^+}, X_{S_j}\right)$, $\delta = 3$, $\sigma = 1.1$, $\alpha = 0.95$ and $\tau = 4$. We first sketch in Figures 2a and 2b the shapes of $\bar{F}_{R_{T_{j,k}}|X_{T_{j,k}},X_{E_j^+}}\left(r \mid x_{j,k}, x_j\right)$ and $S\left(R_{T_{j,k}} \mid X_{T_{j,k}} = x_{j,k}, X_{E_j^+} = x_j\right)$ respectively. Obviously, they are decreasing with respect to $x_{j,k}$ and x_j, hence easily controllable and more informative than the sole



(a) $\bar{F}_{R_{T_{j,k}}|X_{T_{j,k}},X_{E_j^+}}\left(r \mid x_{j,k}, x_j\right)$ (b) $S\left(R_{T_{j,k}} \mid X_{T_{j,k}} = x_{j,k}, X_{E_j^+} = x_j\right)$

Figure 2: Shapes of the sf and std of the system RUL

$X_{T_{j,k}}$ or $X_{E_j^+}$. We next represent the schematic behavior of the maintained system and the associated maintenance actions in Figure 3. For a specific explanation, let us describe what happens on the system during the fourth maintenance cycle $\left[E_3^+, E_4^+\right]$ (see Figure 3a), the system behavior and the maintenance actions over other cycles are similar. As shown in Figure 3b, the std of the system RUL at the three first inspection times of the cycle is still greater than the threshold σ, so further inspections with period δ are needed. At the fourth inspection time $T_{3,4}$, the RUL std $S\left(R_{T_{3,4}} \mid X_{T_{3,4}}, X_{E_3^+}\right) < \sigma$, no more inspection is thus required, and a maintenance action is planned $\psi\left(\alpha, X_{T_{3,4}}, X_{E_3^+}\right)$ time unit later as depicted in Figure 3c. At the maintenance time $S_3 = T_{3,4} + \psi\left(\alpha, X_{T_{3,4}}, X_{E_3^+}\right)$, the system degradation level is still smaller than the failure threshold L (see Figure 3a), so a preventive action is immediately carried out on the running system. Since the duration required for a repair $\rho\left(X_{E_3^+}, X_{S_3}\right)$ is less than the threshold τ, the maintenance action



(a) System degradation state and maintenance actions

(b) RUL sdt at inspection times

(c) Waiting time before the beginning of maintenance actions

(d) Maintenance duration

Figure 3: Schematic behavior of the maintained system under the policy $(\delta, \sigma, \alpha, \tau)$

is an IR instead of a PR. During the IR, the system is inactivated. After $\rho\left(X_{E_3^+}, X_{S_3}\right)$ time unit, the system degradation is returned to a level $X_{E_4^+} \sim g\left(x_4 \mid X_{E_3^+}, X_{S_3}\right)$, and the maintenance cycle ends. We can find that the fixed thresholds σ and τ in Figures 3b and 3d correspond to variable degradation thresholds x_σ and x_τ in Figure 3a (see also Subsection 3.1 for the definition of x_σ and x_τ). Besides, a single threshold α can match with various waiting times depending on the system degradation state (see Figure 3c). This high flexibility confirms the versatility of the $(\delta, \sigma, \alpha, \tau)$ policy without a large number of decision variables.

3. Probabilistic behavior of the maintained system at steady state

Evaluation of the cost rate C_∞ requires the knowledge about the probabilistic degradation behavior of the maintained system at steady state [10]. In this section, we take advantage of the semi-regenerative properties of the degradation process $\{X_t\}_{t\geq 0}$ to derive such a stationary measure. Indeed, given $X_{E_j^+}$ returned by a repair or replacement at time E_j, $j \in \mathbb{N}$, the future $\{X_t\}_{t>E_j}$ of the degradation process $\{X_t\}_{t\geq 0}$ is conditionally independent of its past $\{X_t\}_{0\leq t<E_j}$. Following [54], $\{X_t\}_{t\geq 0}$ appears as a semi-regenerative process with an embedded Markov renewal process $\left\{Y_j, E_j^+\right\}_{j\in\mathbb{N}}$, where $Y_j = X_{E_j^+}$ denotes the system degradation level at the semi-regenerative time E_j^+. The Markov chain $\left\{Y_j\right\}_{j\in\mathbb{N}}$ starts from $Y_0 = 0$, takes the value in the continuous state space $[0, L)$, and comes back to 0 (i.e., a regeneration set) almost surely due to replacement actions. According to [55, Chapter VII, Section 3], $\left\{Y_j\right\}_{j\in\mathbb{N}}$ is Harris recurrent and has an unique stationary measure π which is the solution of the following invariance equation [56, Chapter 10]

$$\pi\left(\mathrm{d}x_{j+1}\right) = \int_{[0,L)} P\left(x_j, \mathrm{d}x_{j+1}\right) \pi\left(\mathrm{d}x_j\right), \tag{15}$$

where $P\left(x_j, \mathrm{d}x_{j+1}\right)$ stands for the transition pdf of $\left\{Y_j\right\}_{j\in\mathbb{N}}$ from $X_{E_j^+} = x_j$ to $X_{E_{j+1}^+} = x_{j+1}$. We seek hereinafter a closed-form expression of $P\left(x_j, \mathrm{d}x_{j+1}\right)$, and thence a solution for (15).

3.1. *Transition probability density function* $P\left(x_j, \mathrm{d}x_{j+1}\right)$

The main difficulty in formulating $P\left(x_j, \mathrm{d}x_{j+1}\right)$ is the difference in the nature of the decision thresholds α and τ. To overcome this obstacle, we first convert all these thresholds into the corresponding degradation levels $x_\sigma\left(x_j\right)$ and $x_\tau\left(x_j\right)$ such that

$$x_\sigma\left(x_j\right) = \inf\left\{x_j \leq x_{j,k} \leq L : S\left(R_{T_{j,k}} \mid X_{T_{j,k}} = x_{j,k}, X_{E_j^+} = x_j\right) \leq \sigma\right\}, \tag{16}$$

and

$$x_\tau\left(x_j\right) = \min\left\{L, \inf\left\{x_s \geq x_j : \rho\left(X_{E_j^+} = x_j, X_{S_j} = x_s\right) \geq \tau\right\}\right\}, \tag{17}$$

where $S\left(R_{T_{j,k}} \mid X_{T_{j,k}} = x_{j,k}, X_{E_j^+} = x_j\right)$ and $\rho\left(X_{E_j^+} = x_j, X_{S_j} = x_s\right)$ are given from (9) and (10) respectively. Next, we exhaustively analyze all maintenance scenarios over $\left[E_j^+, E_{j+1}^+\right]$ based on the system degradation, and effectuate associated probabilistic computations. Consequently, we obtain the closed-form expression of $P\left(x_j, \mathrm{d}x_{j+1}\right)$ as

$$P\left(x_j, \mathrm{d}x_{j+1}\right) = \delta_0\left(\mathrm{d}x_{j+1}\right) \cdot p\left(x_j\right) + f\left(x_{j+1} \mid x_j\right) \cdot \mathrm{d}x_{j+1}, \tag{18}$$

where $\delta_0(\cdot)$ stands for the Dirac mass at 0, $p\left(x_j\right)$ denotes the probability that the system is AGAN due to a PR or CR

$$\begin{aligned} p\left(x_j\right) = 1_{\{x_\tau(x_j) \leq x_\sigma(x_j)\}} \cdot 1_{\{0 \leq x_j \leq L\}} + \left(p_1\left(x_j\right) + p_2\left(x_j\right) + p_3\left(x_j\right) + p_4\left(x_j\right)\right) \cdot 1_{\{0 \leq x_j < x_\sigma(x_j) < x_\tau(x_j)\}} \\ + p_5\left(x_j\right) \cdot 1_{\{x_\sigma(x_j) \leq x_j < x_\tau(x_j)\}} + 1_{\{x_\sigma(x_j) < x_\tau(x_j) \leq x_j < L\}}, \end{aligned} \tag{19}$$

and $f\left(x_{j+1} \mid x_j\right)$ is the pdf of the system degradation state after an IR

$$f\left(x_{j+1} \mid x_j\right) = \left(f_1\left(x_{j+1} \mid x_j\right) + f_2\left(x_{j+1} \mid x_j\right)\right) \cdot 1_{\{0 \leq x_j < x_\sigma(x_j) < x_\tau(x_j)\}} + f_3\left(x_{j+1} \mid x_j\right) \cdot 1_{\{x_\sigma(x_j) \leq x_j \leq x_\tau(x_j)\}}. \tag{20}$$

In (19), $p_1\left(x_j\right)$, $p_2\left(x_j\right)$, $p_3\left(x_j\right)$, $p_4\left(x_j\right)$ and $p_5\left(x_j\right)$ are respectively the transition probabilities conditional on $X_{E_j^+} = x_j$ such that

- a PR or a CR starts after 1 inspection period (with $X_{T_{j,1}} \geq x_\tau\left(x_j\right)$)

$$p_1\left(x_j\right) = \bar{F}\left(x_\tau\left(x_j\right) - x_j; \mu\left(x_j\right) \cdot \delta, \lambda \cdot \delta^2\right), \tag{21}$$

- a PR or a CR starts after 1 inspection period (with $x_\sigma\left(x_j\right) \leq X_{T_{j,1}} < x_\tau\left(x_j\right)$)

$$p_2\left(x_j\right) = \int_{x_\sigma(x_j)}^{x_\tau(x_j)} \bar{F}\left(x_\tau\left(x_j\right) - w; \mu\left(x_j\right) \cdot \psi\left(\alpha, w, x_j\right), \lambda \cdot \psi^2\left(\alpha, w, x_j\right)\right) \cdot f\left(w - x_j; \mu\left(x_j\right) \cdot \delta, \lambda \cdot \delta^2\right) \mathrm{d}w, \tag{22}$$

- a PR or a CR starts after at least 2 inspection periods (with $X_{T_{j,k+1}} \geq x_\tau\left(x_j\right)$, $k = 1, 2, \ldots$)

$$p_3\left(x_j\right) = \int_{x_j}^{x_\sigma(x_j)} \bar{F}\left(x_\tau\left(x_j\right) - w; \mu\left(x_j\right) \cdot \delta, \lambda \cdot \delta^2\right) \cdot \sum_{k=1}^{\infty} f\left(w - x_j; \mu\left(x_j\right) \cdot k\delta, \lambda \cdot (k\delta)^2\right) \mathrm{d}w, \tag{23}$$

- a PR or a CR starts after at least 2 inspection periods (with $x_\sigma\left(x_j\right) \leq X_{T_{j,k+1}} < x_\tau\left(x_j\right)$, $k = 1, 2, \ldots$)

$$\begin{aligned} p_4\left(x_j\right) = \int_{x_\sigma(x_j)}^{x_\tau(x_j)} \bar{F}\left(x_\tau\left(x_j\right) - z; \mu\left(x_j\right) \cdot \psi\left(\alpha, z, x_j\right), \lambda \cdot \psi^2\left(\alpha, z, x_j\right)\right) \\ \times \left(\int_{x_j}^{x_\sigma(x_j)} f\left(z - w; \mu\left(x_j\right) \cdot \delta, \lambda \cdot \delta^2\right) \cdot \sum_{k=1}^{\infty} f\left(w - x_j; \mu\left(x_j\right) \cdot k\delta, \lambda \cdot (k\delta)^2\right) \mathrm{d}w\right) \mathrm{d}z, \end{aligned} \tag{24}$$

- a PR or a CR starts without any inspection before

$$p_5\left(x_j\right) = \bar{F}\left(x_\tau\left(x_j\right) - x_j; \mu\left(x_j\right) \cdot \psi\left(\alpha, x_j, x_j\right), \lambda \cdot \psi^2\left(\alpha, x_j, x_j\right)\right), \tag{25}$$

while in (20), $f_1\left(x_{j+1} \mid x_j\right)$, $f_2\left(x_{j+1} \mid x_j\right)$ and $f_3\left(x_{j+1} \mid x_j\right)$ are the transition pdf conditional on $X_{E_j^+} = x_j$ such that

- an IR starts after 1 inspection period

$$f_1\left(x_{j+1}\mid x_j\right)=\int_{x_\sigma(x_j)}^{x_\tau(x_j)}1_{\{x_{j+1}\neq 0,x_j\leq x_{j+1}\leq z\}}\cdot g\left(x_{j+1}\mid x_j,z\right)$$
$$\times\left(\int_{x_\sigma(x_j)}^{x_\tau(x_j)}1_{\{w<z\}}\cdot f\left(z-w;\mu\left(x_j\right)\cdot\psi\left(\alpha,w,x_j\right),\lambda\cdot\psi^2\left(\alpha,w,x_j\right)\right)\cdot f\left(w-x_j;\mu\left(x_j\right)\cdot\delta,\lambda\cdot\delta^2\right)\mathrm{d}w\right)\mathrm{d}z, \quad (26)$$

- an IR starts after at least 2 inspection periods

$$f_2\left(x_{j+1}\mid x_j\right)=\int_{x_\sigma(x_j)}^{x_\tau(x_j)}1_{\{x_{j+1}\neq 0,x_j\leq x_{j+1}\leq s\}}\cdot g\left(x_{j+1}\mid x_j,s\right)\cdot\left[\int_{x_\sigma(x_j)}^{x_\tau(x_j)}1_{\{z<s\}}\cdot f\left(s-z;\mu\left(x_j\right)\cdot\psi\left(\alpha,z,x_j\right),\lambda\cdot\psi^2\left(\alpha,z,x_j\right)\right)\right.$$
$$\left.\times\left(\int_{x_j}^{x_\sigma(x_j)}f\left(z-w;\mu\left(x_j\right)\cdot\delta,\lambda\cdot\delta^2\right)\cdot\sum_{k=1}^{\infty}f\left(w-x_j;\mu\left(x_j\right)\cdot k\delta,\lambda\cdot(k\delta)^2\right)\mathrm{d}w\right)\mathrm{d}z\right]\mathrm{d}s, \quad (27)$$

- an IR starts without any inspection before

$$f_3\left(x_{j+1}\mid x_j\right)=\int_{x_j}^{x_\tau(x_j)}1_{\{x_{j+1}\neq 0,x_j\leq x_{j+1}\leq w\}}\cdot g\left(x_{j+1}\mid x_j,w\right)\cdot f\left(w-x_j;\mu\left(x_j\right)\cdot\psi\left(\alpha,x_j,x_j\right),\lambda\cdot\psi^2\left(\alpha,x_j,x_j\right)\right)\mathrm{d}w, \quad (28)$$

where $f(\cdot;\cdot,\cdot)$ and $\bar{F}(\cdot;\cdot,\cdot)$ are derived from (3) and (5).

3.2. *Stationary probability density function $\pi\left(\mathrm{d}x_{j+1}\right)$*

From the closed-form expression (18) of $P\left(x_j,\mathrm{d}x_{j+1}\right)$, we can derive the stationary pdf $\pi\left(\mathrm{d}x_{j+1}\right)$ as a convex combination of Dirac mass function and a continuous pdf

$$\pi\left(\mathrm{d}x_{j+1}\right)=a\cdot\delta_0\left(\mathrm{d}x_{j+1}\right)+(1-a)\cdot b\left(x_{j+1}\right)\cdot 1_{\{0<x_{j+1}<L\}}\mathrm{d}x_{j+1}, \quad (29)$$

where

$$a=\frac{1}{1+\int_0^L B\left(x_{j+1}\right)\mathrm{d}x_{j+1}} \quad \text{and} \quad b\left(x_{j+1}\right)=\frac{a}{(1-a)}\cdot B\left(x_{j+1}\right). \quad (30)$$

The value of a has to belongs to $[0,1]$ as it is a probability. When $a=0$, the system is never replaced at the end of a maintenance cycle; on the contrary, a replacement is always performed when $a=1$. To obtain the values of $B\left(x_{j+1}\right)$, we divide the value interval $(0,L)$ of x_{j+1} by N sub-intervals with same length $h=\frac{L}{N}$: $x_{j+1,0}=0, x_{j+1,N}=L$, and $x_{j+1,m}=m\cdot h$ with $m=1,\ldots,N$. Next, we approximate $B\left(x_{j+1}\right)$ with $0<x_{j+1}<L$ by $B_m=B\left(x_{j+1,m}\right)$ with $m=1,\ldots,N$, such that

$$\mathbf{B}=\mathbf{M}^{-1}\cdot\mathbf{q}, \quad (31)$$

where $\mathbf{B}=\begin{bmatrix}B_0 & \cdots & B_N\end{bmatrix}^T$, $\mathbf{q}=\begin{bmatrix}q_0 & \cdots & q_N\end{bmatrix}^T$, and $\mathbf{M}=(M_{m,n})_{0\leq m,n\leq N}$ is a lower triangular matrix with

$$\begin{aligned}
M_{m,m} &= 1-\frac{h}{2}K_{m,m}, && m=1,\ldots,N,\\
M_{m,n} &= -hK_{m,n}, && 1\leq n\leq m\leq N,\\
M_{m,0} &= -\frac{h}{2}K_{m,0}, && m=1,\ldots,N,\\
M_{0,0} &= 1.
\end{aligned}$$

The quantities $q_m=q\left(x_{j+1,m}\right)$ and $K_{m,n}=K\left(x_{j+1,m}\mid x_{j,n}\right)$, with $n=1,\ldots,m$, are given by

$$q\left(x_{j+1}\right)=\left(f_1\left(x_{j+1}\mid 0\right)+f_2\left(x_{j+1}\mid 0\right)\right)\cdot 1_{\{x_{j+1}\leq x_\tau(0)\}}\cdot 1_{\{x_\sigma(0)<x_\tau(0)\}}, \quad (32)$$

and

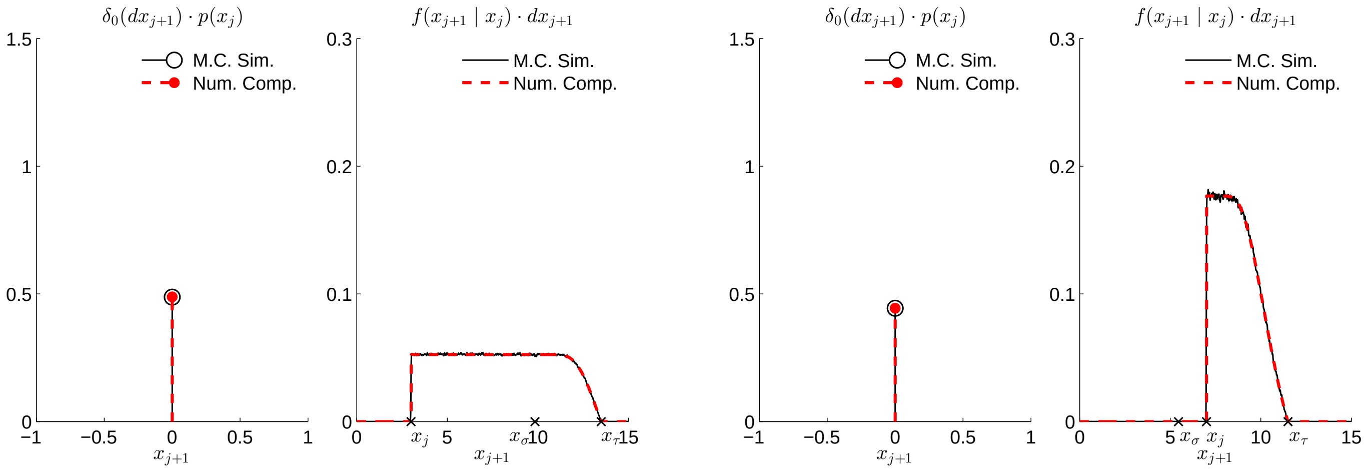
$$K\left(x_{j+1}\mid x_j\right)=\left(f_1\left(x_{j+1}\mid x_j\right)+f_2\left(x_{j+1}\mid x_j\right)\right)\cdot 1_{\{0\leq x_j<x_\sigma(x_j)<x_\tau(x_j)\}}+f_3\left(x_{j+1}\mid x_j\right)\cdot 1_{\{x_\sigma(x_j)\leq x_j\leq x_\tau(x_j)\}}, \quad (33)$$

where $f_1\left(x_{j+1}\mid x_j\right)$, $f_2\left(x_{j+1}\mid x_j\right)$ and $f_3\left(x_{j+1}\mid x_j\right)$ are given from (26), (27) and (28).

3.3. Numerical illustration

Based on the maintained system defined by $L = 15$, $\mu\left(X_{E_j^+}\right) = 1 + 0.1 \cdot X_{E_j^+}$, $\lambda = 4$, $\rho\left(X_{E_j^+}, X_{S_j}\right) = \rho_0 + \rho_1\left(X_{E_j^+}, X_{S_j}\right) = 1 + 0.1 \cdot X_{E_j^+} + 0.2 \cdot X_{S_j}$ and a continuous uniform pdf for $g\left(x_{j+1} \mid X_{E_j^+}, X_{S_j}\right)$, our goal is twofold: (*i*) show the shapes of $P\left(x_j, \mathrm{d}x_{j+1}\right)$ and of $\pi\left(\mathrm{d}x_{j+1}\right)$, and (*ii*) validate their mathematical development.

To illustrate both Dirac part and continuous part of $P\left(x_j, \mathrm{d}x_{j+1}\right)$, we consider two configurations: $\left\{0 \le x_j < x_\sigma\left(x_j\right) < x_\tau\left(x_j\right)\right\}$: $x_j = 3$, $\delta = 3$, $\sigma = 1.1$, $\alpha = 0.95$, $\tau = 4$, and $\left\{x_\sigma\left(x_j\right) \le x_j \le x_\tau\left(x_j\right)\right\}$: $x_j = 7$, $\delta = 3$, $\sigma = 1.5$, $\alpha = 0.95$, $\tau = 4$. The others (i.e., $\left\{x_\tau\left(x_j\right) \le x_\sigma\left(x_j\right), 0 \le x_j \le L\right\}$ and $\left\{x_\sigma\left(x_j\right) < x_\tau\left(x_j\right) \le x_j < L\right\}$), which merely correspond to a Dirac measure at 0 with magnitude 1, are omitted. We next use two different methods to sketch out the shape of $P\left(x_j, \mathrm{d}x_{j+1}\right)$ in Figures 4a and 4b. The solid black curves



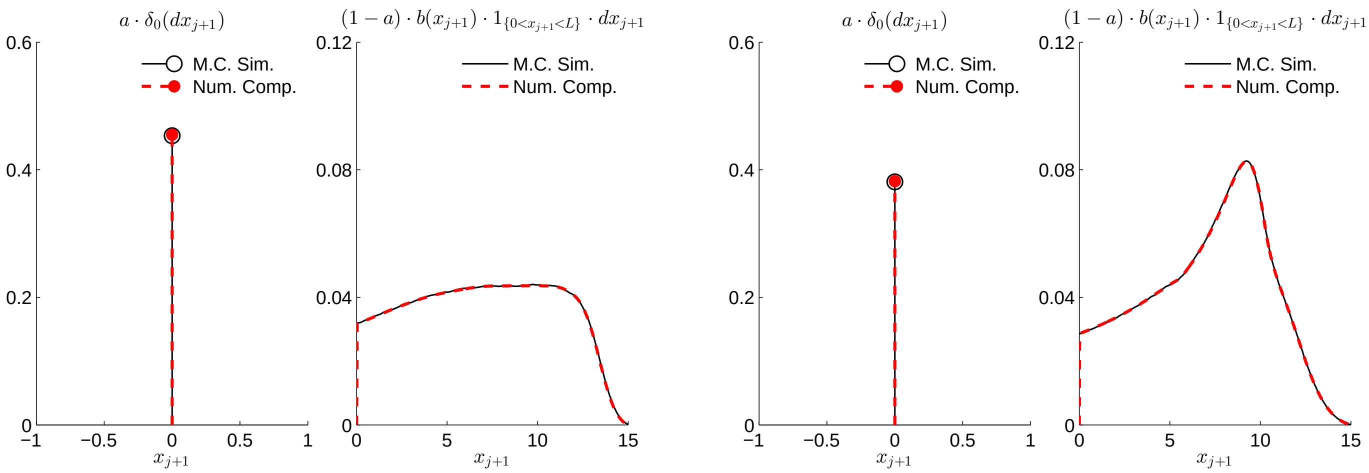
(a) $x_j = 3, \delta = 3, \sigma = 1.1, \alpha = 0.95, \tau = 4$

(b) $x_j = 7, \delta = 3, \sigma = 1.5, \alpha = 0.95, \tau = 4$

Figure 4: Shapes of the transition pdf $P\left(x_j, \mathrm{d}x_{j+1}\right)$

are obtained by combining the Monte Carlo simulation (following the flowchart in Figure 1) and the kernel density estimation (KDE) [57]. Meanwhile, the dashed red curves are returned by the numerical computation of (18) using the well-known trapezoid rule. The identical results confirm the exactness of the mathematical formulation for $P\left(x_j, \mathrm{d}x_{j+1}\right)$.

Applying the above methods to the two following configurations of the $(\delta, \sigma, \alpha, \tau)$ policy: $\delta = 3$, $\sigma = 1.1$, $\alpha = 0.95$, $\tau = 4$, and $\delta = 3$, $\sigma = 1.5$, $\alpha = 0.95$, $\tau = 4$, we obtain in Figures 5a and 5b the shapes of $\pi\left(\mathrm{d}x_{j+1}\right)$. Once again, the identical results given by



(a) $\delta = 3$, $\sigma = 1.1$, $\alpha = 0.95$ and $\tau = 4$

(b) $\delta = 3$, $\sigma = 1.5$, $\alpha = 0.95$ and $\tau = 4$

Figure 5: Shapes of the stationary pdf $\pi\left(\mathrm{d}x_{j+1}\right)$

both the numerical computation of (29) (i.e., dashed red curves) and the combination of Monte Carlo simulation and of KDE (i.e., solid black curves) justify the correctness of the mathematical development of $\pi\left(\mathrm{d}x_{j+1}\right)$.

4. Long-run maintenance cost rate evaluation and maintenance policy optimization

The aim of this section is to evaluate the long-run maintenance cost rate (13) using the semi-regenerative technique [40]

$$C_{\infty}(\delta,\sigma,\alpha,\tau)=\lim_{t\to\infty}\frac{C(t)}{t}=\frac{E_{\pi}\left[C\left(\left[E_j^+,E_{j+1}^+\right]\right)\right]}{E_{\pi}\left[\Delta E_{j+1}\right]},\tag{34}$$

where $C(t)$ is given from (12), ΔE_{j+1} denotes the length of the $(k+1)$-st Markov renewal cycle $\left[E_j^+,E_{j+1}^+\right]$ with $X_{E_j^+}=x_j\in[0,L)$, $E_{\pi}[\cdot]$ stands for the s-expectation with respect to the stationary measure π. Once the closed-form expression of $C_{\infty}(\delta,\sigma,\alpha,\tau)$ is available, we apply the generalized pattern search algorithm [58] to search the optimal configuration $\left(\delta_{opt},\sigma_{opt},\alpha_{opt},\tau_{opt}\right)$ of the $(\delta,\sigma,\alpha,\tau)$ policy.

4.1. Long-run maintenance cost rate evaluation

From (12), we can express (34) as

$$\begin{aligned}C_{\infty}(\delta,\sigma,\alpha,\tau)=\frac{1}{E_{\pi}\left[\Delta E_{j+1}\right]}\cdot\Big(&C_m\cdot E_{\pi}\left[N_m\left(\left[E_j^+,E_{j+1}^+\right]\right)\right]+C_r\cdot E_{\pi}\left[N_r\left(\left[E_j^+,E_{j+1}^+\right]\right)\right]\\&+C_o\cdot E_{\pi}\left[N_o\left(\left[E_j^+,E_{j+1}^+\right]\right)\right]+C_i\cdot E_{\pi}\left[I\left(\left[E_j^+,E_{j+1}^+\right]\right)\right]+C_u\cdot E_{\pi}\left[U\left(\left[E_j^+,E_{j+1}^+\right]\right)\right]\Big),\end{aligned}\tag{35}$$

where $N_m\left(\left[E_j^+,E_{j+1}^+\right]\right)$, $N_r\left(\left[E_j^+,E_{j+1}^+\right]\right)$ and $N_o\left(\left[E_j^+,E_{j+1}^+\right]\right)$ are the number of inspections, of repair and replacement over $\left[E_j^+,E_{j+1}^+\right]$, $I\left(\left[E_j^+,E_{j+1}^+\right]\right)$ and $U\left(\left[E_j^+,E_{j+1}^+\right]\right)$ are the associated duration of system inactivity and unavailability. Since there is one and only one maintenance action (either PR, CR or IR) over $\left[E_j^+,E_{j+1}^+\right]$,

$$E_{\pi}\left[N_r\left(\left[E_j^+,E_{j+1}^+\right]\right)\right]+E_{\pi}\left[N_o\left(\left[E_j^+,E_{j+1}^+\right]\right)\right]=1.\tag{36}$$

Moreover, we can express $\left[E_j^+,E_{j+1}^+\right]$ as $\left[E_j^+,E_j^++\delta\cdot N_m\left(\left[E_j^+,E_{j+1}^+\right]\right)\right]\cup\left[E_j^++\delta\cdot N_m\left(\left[E_j^+,E_{j+1}^+\right]\right),S_j\right]\cup\left[S_j,E_{j+1}^+\right]$, so

$$\Delta E_{j+1}=\delta\cdot N_m\left(\left[E_j^+,E_{j+1}^+\right]\right)+W\left(\left[E_j^+,E_{j+1}^+\right]\right)+I\left(\left[E_j^+,E_{j+1}^+\right]\right),\tag{37}$$

where $W\left(\left[E_j^+,E_{j+1}^+\right]\right)$ denotes the waiting duration before a maintenance starts since the last inspection over $\left[E_j^+,E_{j+1}^+\right]$. Consequently, the cost rate (35) is rewritten by

$$\begin{aligned}C_{\infty}(\delta,\sigma,\alpha,\tau)=\frac{1}{\delta\cdot E_{\pi}\left[N_m\left(\left[E_j^+,E_{j+1}^+\right]\right)\right]+E_{\pi}\left[W\left(\left[E_j^+,E_{j+1}^+\right]\right)\right]+E_{\pi}\left[I\left(\left[E_j^+,E_{j+1}^+\right]\right)\right]}\cdot\Big(C_m\cdot E_{\pi}\left[N_m\left(\left[E_j^+,E_{j+1}^+\right]\right)\right]\\+C_o-(C_o-C_r)\cdot E_{\pi}\left[N_r\left(\left[E_j^+,E_{j+1}^+\right]\right)\right]+C_i\cdot E_{\pi}\left[I\left(\left[E_j^+,E_{j+1}^+\right]\right)\right]+C_u\cdot E_{\pi}\left[U\left(\left[E_j^+,E_{j+1}^+\right]\right)\right]\Big).\end{aligned}\tag{38}$$

In the following, we give the mathematical expressions of $E_{\pi}\left[N_m\left(\left[E_j^+,E_{j+1}^+\right]\right)\right]$, $E_{\pi}\left[N_r\left(\left[E_j^+,E_{j+1}^+\right]\right)\right]$, $E_{\pi}\left[W\left(\left[E_j^+,E_{j+1}^+\right]\right)\right]$, $E_{\pi}\left[I\left(\left[E_j^+,E_{j+1}^+\right]\right)\right]$ and $E_{\pi}\left[U\left(\left[E_j^+,E_{j+1}^+\right]\right)\right]$.

4.1.1. Expected number of inspections over $\left[E_j^+,E_{j+1}^+\right]$

Starting at $X_{E_j^+}\in[0,L)$, the cycle $\left[E_j^+,E_{j+1}^+\right]$ may be stopped (i) after 1 inspection if $0\le X_{E_j^+}<x_{\sigma}\left(X_{E_j^+}\right)\le X_{T_{j,1}}$, ($ii$) after $k+1$ with $k\in\mathbb{N}^*$ if $0\le X_{E_j^+}<X_{T_{j,k}}<x_{\sigma}\left(X_{E_j^+}\right)\le X_{T_{j,k+1}}$, or even ($iii$) without inspection if $x_{\sigma}\left(X_{E_j^+}\right)<X_{E_j^+}<L$. So, the number of inspections over $\left[E_j^+,E_{j+1}^+\right]$ can be expressed as

$$N_m\left(\left[E_j^+,E_{j+1}^+\right]\right)=1_{\left\{0\le X_{E_j^+}<x_{\sigma}\left(X_{E_j^+}\right)\le X_{T_{j,1}}\right\}}+\sum_{k=1}^{\infty}(k+1)\cdot 1_{\left\{0\le X_{E_j^+}<X_{T_{j,k}}<x_{\sigma}\left(X_{E_j^+}\right)\le X_{T_{j,k+1}}\right\}}.\tag{39}$$

Its expected value with respect to the stationary measure π is thus computed by

$$E_{\pi}\left[N_m\left(\left[E_j^+,E_{j+1}^+\right]\right)\right]=P_{m,1}+\sum_{k=1}^{\infty}(k+1)\cdot P_{m,k+1},\tag{40}$$

where

- $P_{m,1}$ denotes the probability that only one inspection is performed over $\left[E_j^+, E_{j+1}^+\right]$

$$P_{m,1} = a \cdot \bar{F}\left(x_\sigma(0); \mu(0)\cdot\delta, \lambda\cdot\delta^2\right) + (1-a)\cdot\int_0^{x_\sigma(x_j)} \bar{F}\left(x_\sigma(x_j) - x_j; \mu(x_j)\cdot\delta, \lambda\cdot\delta^2\right)\cdot b(x_j)\,\mathrm{d}x_j, \tag{41}$$

- $P_{m,k+1}$ denotes the probability that $(k+1)$ inspections, with $k \in \mathbb{N}^*$, are performed over $\left[E_j^+, E_{j+1}^+\right]$

$$\begin{aligned} P_{m,k+1} = a \cdot \int_0^{x_\sigma(0)} \bar{F}\left(x_\sigma(0) - w; \mu(0)\cdot\delta, \lambda\cdot\delta^2\right)\cdot f\left(w; \mu(0)\cdot k\delta, \lambda\cdot(k\delta)^2\right)\mathrm{d}w + (1-a)\times \\ \int_0^{x_\sigma(x_j)}\left(\int_{x_j}^{x_\sigma(x_j)} \bar{F}\left(x_\sigma(x_j) - w; \mu(x_j)\cdot\delta, \lambda\cdot\delta^2\right)\cdot f\left(w - x_j; \mu(x_j)\cdot k\delta, \lambda\cdot(k\delta)^2\right)\mathrm{d}w\right)\cdot b(x_j)\,\mathrm{d}x_j, \end{aligned} \tag{42}$$

with a and $b(x_j)$ given from (30).

4.1.2. Expected number of repair over $\left[E_j^+, E_{j+1}^+\right]$

Since we can perform an IR over $\left[E_j^+, E_{j+1}^+\right]$ (*i*) without inspection if $x_\sigma\left(X_{E_j^+}\right) \le X_{E_j^+} \le X_{S_j} < x_\tau\left(X_{E_j^+}\right)$, (*ii*) after 1 inspection if $0 \le X_{E_j^+} < x_\sigma\left(X_{E_j^+}\right) \le X_{T_{j,1}} \le X_{S_j} < x_\tau\left(X_{E_j^+}\right)$, or (*iii*) after $(k+1)$ inspections, with $k \in \mathbb{N}^*$, if $0 \le X_{E_j^+} < X_{T_{j,k}} < x_\sigma\left(X_{E_j^+}\right) \le X_{T_{j,k+1}} \le X_{S_j} < x_\tau\left(X_{E_j^+}\right)$, the number of IR over $\left[E_j^+, E_{j+1}^+\right]$ can be expressed by

$$N_r\left(\left[E_j^+, E_{j+1}^+\right]\right) = 1_{\left\{x_\sigma\left(X_{E_j^+}\right) \le X_{E_j^+} \le X_{S_j} < x_\tau\left(X_{E_j^+}\right)\right\}} + 1_{\left\{0 \le X_{E_j^+} < x_\sigma\left(X_{E_j^+}\right) \le X_{T_{j,1}} \le X_{S_j} < x_\tau\left(X_{E_j^+}\right)\right\}} + \sum_{k=1}^{\infty} 1_{\left\{0 \le X_{E_j^+} < X_{T_{j,k}} < x_\sigma\left(X_{E_j^+}\right) \le X_{T_{j,k+1}} \le X_{S_j} < x_\tau\left(X_{E_j^+}\right)\right\}}. \tag{43}$$

This leads us to compute $E_\pi\left[N_r\left(\left[E_j^+, E_{j+1}^+\right]\right)\right]$ as

$$E_\pi\left[N_r\left(\left[E_j^+, E_{j+1}^+\right]\right)\right] = P_{r,0} + P_{r,1} + \sum_{k=1}^{\infty} P_{r,k+1}, \tag{44}$$

where

- $P_{r,0}$ denotes the probability that an IR is performed over $\left[E_j^+, E_{j+1}^+\right]$ without any inspection

$$P_{r,0} = (1-a)\cdot\int_{x_\sigma(x_j)}^{x_\tau(x_j)} F\left(x_\tau(x_j) - x_j; \mu(x_j)\cdot\psi(\alpha, x_j, x_j), \lambda\cdot\psi^2(\alpha, x_j, x_j)\right)\cdot b(x_j)\,\mathrm{d}x_j, \tag{45}$$

- $P_{r,1}$ denotes the probability that an IR is performed over $\left[E_j^+, E_{j+1}^+\right]$ after one inspection

$$\begin{aligned} P_{r,1} = a \cdot \int_{x_\sigma(0)}^{x_\tau(0)} F\left(x_\tau(0) - w; \mu(0)\cdot\psi(\alpha, w, 0), \lambda\cdot\psi^2(\alpha, w, 0)\right)\cdot f\left(w; \mu(0)\cdot\delta, \lambda\cdot\delta^2\right)\mathrm{d}w + (1-a)\cdot\int_0^{x_\sigma(x_j)} b(x_j) \\ \times\left(\int_{x_\sigma(x_j)}^{x_\tau(x_j)} F\left(x_\tau(x_j) - w; \mu(x_j)\cdot\psi(\alpha, w, x_j), \lambda\cdot\psi^2(\alpha, w, x_j)\right)\cdot f\left(w - x_j; \mu(x_j)\cdot\delta, \lambda\cdot\delta^2\right)\mathrm{d}w\right)\mathrm{d}x_j, \end{aligned} \tag{46}$$

- $P_{r,k+1}$ denotes the probability that an IR is performed over $\left[E_j^+, E_{j+1}^+\right]$ after $(k+1)$ inspections with $k \in \mathbb{N}^*$

$$\begin{aligned} P_{r,k+1} = a \cdot \int_0^{x_\sigma(0)}\left(\int_{x_\sigma(0)}^{x_\tau(0)} F\left(x_\tau(0) - z; \mu(0)\cdot\psi(\alpha, z, 0), \lambda\cdot\psi^2(\alpha, z, 0)\right)\right. \\ \left.\times f\left(z - w; \mu(0)\cdot\delta, \lambda\cdot\delta^2\right)\mathrm{d}z\right)\cdot f\left(w; \mu(0)\cdot k\delta, \lambda\cdot(k\delta)^2\right)\mathrm{d}w \\ + (1-a)\cdot\int_0^{x_\sigma(x_j)}\left(\int_{x_j}^{x_\sigma(x_j)}\left(\int_{x_\sigma(x_j)}^{x_\tau(x_j)} F\left(x_\tau(x_j) - z; \mu(x_j)\cdot\psi(\alpha, z, x_j), \lambda\cdot\psi^2(\alpha, z, x_j)\right)\right.\right. \\ \left.\left.\times f\left(z - w; \mu(x_j)\cdot\delta, \lambda\cdot\delta^2\right)\mathrm{d}z\right)\cdot f\left(w - x_j; \mu(x_j)\cdot k\delta, \lambda\cdot(k\delta)^2\right)\mathrm{d}w\right)\cdot b(x_j)\,\mathrm{d}x_j, \end{aligned} \tag{47}$$

with a and $b(x_j)$ given from (30).

4.1.3. Expected length of waiting duration over $\left[E_j^+, E_{j+1}^+\right]$

Following the $(\delta, \sigma, \alpha, \tau)$ policy, the considered waiting duration is the time interval from the latest inspection (including the starting time of a Markov renewal cycle) to the beginning of a maintenance action. Since a maintenance can be done without inspection, after one inspection or after $(k+1)$ inspections with $k \in \mathbb{N}^*$, the associated waiting duration over $\left[E_j^+, E_{j+1}^+\right]$ is either (*i*) $\psi\left(\alpha, X_{E_j^+}, X_{E_j^+}\right)$ if $x_\sigma\left(X_{E_j^+}\right) \leq X_{E_j^+} < L$, (*ii*) $\psi\left(\alpha, X_{T_{j,1}}, X_{E_j^+}\right)$ if $0 \leq X_{E_j^+} < x_\sigma\left(X_{E_j^+}\right) \leq X_{T_{j,1}} < L$, or (*iii*) $\psi\left(\alpha, X_{T_{j,k+1}}, X_{E_j^+}\right)$ if $0 \leq X_{E_j^+} < X_{T_{j,k}} < x_\sigma\left(X_{E_j^+}\right) \leq X_{T_{j,k+1}} < L$. We can therefore express the waiting duration $W\left(\left[E_j^+, E_{j+1}^+\right]\right)$ by

$$W\left(\left[E_j^+, E_{j+1}^+\right]\right) = \psi\left(\alpha, X_{E_j^+}, X_{E_j^+}\right) \cdot 1_{\left\{x_\sigma\left(X_{E_j^+}\right) \leq X_{E_j^+} < LL\right\}} + \psi\left(\alpha, X_{T_{j,1}}, X_{E_j^+}\right) \cdot 1_{\left\{0 \leq X_{E_j^+} < x_\sigma\left(X_{E_j^+}\right) \leq X_{T_{j,1}} < L\right\}} + \sum_{k=1}^{\infty} \psi\left(\alpha, X_{T_{j,k+1}}, X_{E_j^+}\right) \cdot 1_{\left\{0 \leq X_{E_j^+} < X_{T_{j,k}} < x_\sigma\left(X_{E_j^+}\right) \leq X_{T_{j,k+1}} < L\right\}}. \quad (48)$$

As such, we obtain its expected value with respect to π as

$$E_\pi\left[W\left(\left[E_j^+, E_{j+1}^+\right]\right)\right] = E_{w,0} + E_{w,1} + \sum_{k=1}^{\infty} E_{w,k+1}, \quad (49)$$

where

- $E_{w,0}$ denotes the expected length of the waiting time just after the beginning of the renewal cycle (i.e., without inspection)

$$E_{w,0} = (1-a) \cdot \int_{x_\sigma(x_j)}^{L} \psi\left(\alpha, x_j, x_j\right) \cdot b\left(x_j\right) \mathrm{d}x_j. \quad (50)$$

- $E_{w,1}$ denotes the expected length of the waiting time just the 1-st inspection

$$E_{w,1} = a \cdot \int_{x_\sigma(0)}^{L} \psi\left(\alpha, w, 0\right) \cdot f\left(w; \mu(0) \cdot \delta, \lambda \cdot \delta^2\right) \mathrm{d}w + (1-a) \cdot \int_0^{x_\sigma(x_j)} \left(\int_{x_\sigma(x_j)}^{L} \psi\left(\alpha, w, x_j\right) \cdot f\left(w - x_j; \mu\left(x_j\right) \cdot \delta, \lambda \cdot \delta^2\right) \mathrm{d}w\right) \cdot b\left(x_j\right) \mathrm{d}x_j, \quad (51)$$

- $E_{w,k+1}$ denotes the expected length of the waiting time just the $(k+1)$-st inspection

$$E_{w,k+1} = a \cdot \int_0^{x_\sigma(0)} \left(\int_{x_\sigma(0)}^{L} \psi\left(\alpha, z, 0\right) \cdot f\left(z - w; \mu(0) \cdot \delta, \lambda \cdot \delta^2\right) \mathrm{d}z\right) \cdot f\left(w; \mu(0) \cdot k\delta, \lambda \cdot (k\delta)^2\right) \mathrm{d}w + (1-a) \cdot \int_0^{x_\sigma(x_j)} \left(\int_{x_j}^{x_\sigma(x_j)} \left(\int_{x_\sigma(x_j)}^{L} \psi\left(\alpha, z, x_j\right) f\left(z - w; \mu\left(x_j\right) \cdot \delta, \lambda \cdot \delta^2\right) \mathrm{d}z\right) \times f\left(w - x_j; \mu\left(x_j\right) \cdot k\delta, \lambda \cdot (k\delta)^2\right) \mathrm{d}w\right) \cdot b\left(x_j\right) \mathrm{d}x_j, \quad (52)$$

with a and $b\left(x_j\right)$ given from (30).

4.1.4. Expected length of the system inactivity duration over $\left[E_j^+, E_{j+1}^+\right]$

During a preventive maintenance (i.e., either a PR or an IR), the considered system is inactivated. Such an inactivity duration over $\left[E_j^+, E_{j+1}^+\right]$ can be obtained by analyzing all possible scenarios of PR or IR

$$I\left(\left[E_j^+, E_{j+1}^+\right]\right) = \rho_0 \cdot 1_{\left\{\text{all scenarios of PR and IR over } \left[E_j^+, E_{j+1}^+\right]\right\}} + \rho_1\left(X_{E_j^+}, X_{S_0}\right) \cdot 1_{\left\{\text{all scenarios of IR over } \left[E_j^+, E_{j+1}^+\right]\right\}}. \quad (53)$$

However, since the scenarios of PR are burdensome to analyze, we propose to base on the ones of CR or IR instead

$$I\left(\left[E_j^+, E_{j+1}^+\right]\right) = \rho_0 \cdot \left(1 - 1_{\left\{\text{all scenarios of CR over } \left[E_j^+, E_{j+1}^+\right]\right\}}\right) + \rho_1\left(X_{E_j^+}, X_{S_0}\right) \cdot 1_{\left\{\text{all scenarios of IR over } \left[E_j^+, E_{j+1}^+\right]\right\}}. \quad (54)$$

All scenarios for IR have been analyzed in Subsection 4.1.2, it is enough to seek the scenarios of CR to make clear (54). Over $\left[E_j^+, E_{j+1}^+\right]$, a CR may be carried out either

1. without inspection if $x_\sigma\left(X_{E_j^+}\right) \leq X_{E_j^+} < L \leq X_{S_0}$,

2. with one inspection: (*i*) at the 1-st inspection if $0 \leq X_{E_j^+} < x_\sigma\left(X_{E_j^+}\right) < L \leq X_{T_{j,1}}$, (*ii*) after the 1-st inspection if $0 \leq X_{E_j^+} < x_\sigma\left(X_{E_j^+}\right) < X_{T_{j,1}} < L \leq X_{S_0}$,
3. or with $k+1$ inspections, $k \in \mathbb{N}^*$: (*i*) at the $(k+1)$-st inspection if $0 \leq X_{E_j^+} < X_{T_{j,k}} < x_\sigma\left(X_{E_j^+}\right) < L \leq X_{T_{j,k+1}}$, (*ii*) after the $(k+1)$-st inspection if $0 \leq X_{E_j^+} < X_{T_{j,k}} < x_\sigma\left(X_{E_j^+}\right) < X_{T_{j,k+1}} < L \leq X_{S_0}$.

Therefore, we can express $I\left(\left[E_j^+, E_{j+1}^+\right]\right)$ as

$$\begin{aligned} I\left(\left[E_j^+, E_{j+1}^+\right]\right) = \rho_0 \Bigg(& 1 - \mathbb{1}_{\left\{x_\sigma\left(X_{E_j^+}\right) \leq X_{E_j^+} < L \leq X_{S_0}\right\}} - \mathbb{1}_{\left\{0 \leq X_{E_j^+} < x_\sigma\left(X_{E_j^+}\right) < L \leq X_{T_{j,1}}\right\}} - \mathbb{1}_{\left\{0 \leq X_{E_j^+} < x_\sigma\left(X_{E_j^+}\right) < X_{T_{j,1}} < L \leq X_{S_0}\right\}} \\ & - \sum_{k=1}^{\infty} \mathbb{1}_{\left\{0 \leq X_{E_j^+} < X_{T_{j,k}} < x_\sigma\left(X_{E_j^+}\right) < L \leq X_{T_{j,k+1}}\right\}} - \sum_{k=1}^{\infty} \mathbb{1}_{\left\{0 \leq X_{E_j^+} < X_{T_{j,k}} < x_\sigma\left(X_{E_j^+}\right) < X_{T_{j,k+1}} < L \leq X_{S_0}\right\}} \Bigg) + \rho_1\left(X_{E_j^+}, X_{S_0}\right) \cdot \mathbb{1}_{\left\{0 \leq x_\sigma\left(X_{E_j^+}\right) \leq X_{E_j^+} \leq X_{S_0} < x_\tau\left(X_{E_j^+}\right)\right\}} \\ & + \rho_1\left(X_{E_j^+}, X_{S_0}\right) \cdot \mathbb{1}_{\left\{0 \leq X_{E_j^+} < x_\sigma\left(X_{E_j^+}\right) \leq X_{T_{j,1}} \leq X_{S_0} < x_\tau\left(X_{E_j^+}\right)\right\}} + \sum_{k=1}^{\infty} \rho_1\left(X_{E_j^+}, X_{S_0}\right) \cdot \mathbb{1}_{\left\{0 \leq X_{E_j^+} < X_{T_{j,k}} < x_\sigma\left(X_{E_j^+}\right) \leq X_{T_{j,k+1}} \leq X_{S_0} < x_\tau\left(X_{E_j^+}\right)\right\}}. \end{aligned} \quad (55)$$

Its expectation with respect to π is computed by

$$E_\pi\left[I\left(\left[E_j^+, E_{j+1}^+\right]\right)\right] = \rho_0 \cdot \left(1 - P_{c,0} - P_{c,1,i} - P_{c,1,s} - \sum_{k=1}^{\infty} P_{c,k+1,i} - \sum_{k=1}^{\infty} P_{c,k+1,s}\right) + E_{i,r,0} + E_{i,r,1} + \sum_{k=1}^{\infty} E_{i,r,k+1}, \quad (56)$$

where

- $P_{c,0}$ denotes the probability that a CR is performed without any inspection

$$P_{c,0} = (1-a) \cdot \int_{x_\sigma(x_j)}^{L} \bar{F}\left(L - x_j; \mu\left(x_j\right) \cdot \psi\left(\alpha, x_j, x_j\right), \lambda \cdot \psi^2\left(\alpha, x_j, x_j\right)\right) \cdot b\left(x_j\right) \mathrm{d}x_j, \quad (57)$$

- $P_{c,1,i}$ denotes the probability that a CR is performed at the 1-st inspection

$$P_{c,1,i} = a \cdot \bar{F}\left(L; \mu(0) \cdot \delta, \lambda \cdot \delta^2\right) + (1-a) \cdot \int_0^{x_\sigma(x_j)} \bar{F}\left(L - x_j; \mu\left(x_j\right) \cdot \delta, \lambda \cdot \delta^2\right) \cdot b\left(x_j\right) \mathrm{d}x_j, \quad (58)$$

- $P_{c,1,s}$ denotes the probability that a CR is performed after the 1-st inspection

$$\begin{aligned} P_{c,1,s} = a \cdot \int_{x_\sigma(0)}^{L} \bar{F}\left(L - w; \mu(0) \cdot \psi(\alpha, w, 0), \lambda \cdot \psi^2(\alpha, w, 0)\right) \cdot f\left(w; \mu(0) \cdot \delta, \lambda \cdot \delta^2\right) \mathrm{d}w + (1-a) \cdot \int_0^{x_\sigma(x_j)} b\left(x_j\right) \times \\ \left(\int_{x_\sigma(x_j)}^{L} \bar{F}\left(L - w; \mu\left(x_j\right) \cdot \psi\left(\alpha, w, x_j\right), \lambda \cdot \psi^2\left(\alpha, w, x_j\right)\right) \cdot f\left(w - x_j; \mu\left(x_j\right) \cdot \delta, \lambda \cdot \delta^2\right) \mathrm{d}w\right) \mathrm{d}x_j, \end{aligned} \quad (59)$$

- $P_{c,k+1,i}$ denotes the probability that a CR is performed at the $(k+1)$-st inspection

$$\begin{aligned} P_{c,k+1,i} = a \cdot \int_0^{x_\sigma(0)} \bar{F}\left(L - w; \mu(0) \cdot \delta, \lambda \cdot \delta^2\right) \cdot f\left(w; \mu(0) \cdot k\delta, \lambda \cdot (k\delta)^2\right) \mathrm{d}w + (1-a) \times \\ \int_0^{x_\sigma(x_j)} \left(\int_{x_j}^{x_\sigma(x_j)} \bar{F}\left(L - w; \mu\left(x_j\right) \cdot \delta, \lambda \cdot \delta^2\right) \cdot f\left(w - x_j; \mu\left(x_j\right) \cdot k\delta, \lambda \cdot (k\delta)^2\right) \mathrm{d}w\right) \cdot b\left(x_j\right) \mathrm{d}x_j, \end{aligned} \quad (60)$$

- $P_{c,k+1,s}$ denotes the probability that a CR is performed after the $(k+1)$-st inspection

$$\begin{aligned} P_{c,k+1,s} = a \cdot \int_0^{x_\sigma(0)} \left(\int_{x_\sigma(0)}^{L} \bar{F}\left(L - z; \mu(0) \cdot \psi(\alpha, z, 0), \lambda \cdot \psi^2(\alpha, z, 0)\right) \cdot f\left(z - w; \mu(0) \cdot \delta, \lambda \cdot \delta^2\right) \mathrm{d}z\right) \\ \times f\left(w; \mu(0) \cdot k\delta, \lambda \cdot (k\delta)^2\right) \mathrm{d}w + (1-a) \cdot \int_0^{x_\sigma(x_j)} \left(\int_{x_j}^{x_\sigma(x_j)} \left(\int_{x_\sigma(x_j)}^{L} \bar{F}\left(L - z; \mu\left(x_j\right) \cdot \psi\left(\alpha, zx_j\right), \lambda \cdot \psi^2\left(\alpha, z, x_j\right)\right)\right.\right. \\ \left.\left. \times f\left(z - w; \mu\left(x_j\right) \cdot \delta, \lambda \cdot \delta^2\right) \mathrm{d}z\right) \cdot f\left(w - x_j; \mu\left(x_j\right) \cdot k\delta, \lambda \cdot (k\delta)^2\right) \mathrm{d}w\right) \cdot b\left(x_j\right) \mathrm{d}x_j, \end{aligned} \quad (61)$$

- $E_{i,r,0}$ denotes the expected length of the system inactivity when no inspection is performed over $\left[E_j^+, E_{j+1}^+\right]$

$$E_{i,r,0} = (1-a)\cdot\int_{x_\sigma(x_j)}^{x_\tau(x_j)} b\left(x_j\right)\left(\int_{x_j}^{x_\tau(x_j)} \rho_1\left(x_j, w\right)\cdot f\left(w - x_j; \mu\left(x_j\right)\cdot\psi\left(\alpha, x_j, x_j\right), \lambda\cdot\psi^2\left(\alpha, x_j, x_j\right)\right)\mathrm{d}w\right)\mathrm{d}x_j, \tag{62}$$

- $E_{i,r,1}$ denotes the expected length of the system inactivity when 1 inspection is performed over $\left[E_j^+, E_{j+1}^+\right]$

$$\begin{aligned} E_{i,r,1} = a\cdot&\int_{x_\sigma(0)}^{x_\tau(0)}\left(\int_{w}^{x_\tau(0)} \rho_1(0,z)\cdot f\left(z-w;\mu(0)\cdot\psi(\alpha,w,0),\lambda\cdot\psi^2(\alpha,w,0)\right)\mathrm{d}z\right)\cdot f\left(w;\mu(0)\cdot\delta,\lambda\cdot\delta^2\right)\mathrm{d}w\\ &+(1-a)\cdot\int_0^{x_\sigma(x_j)}\left(\int_{x_\sigma(x_j)}^{x_\tau(x_j)}\left(\int_w^{x_\tau(x_j)}\rho_1\left(x_j,z\right)\cdot f\left(z-w;\mu\left(x_j\right)\cdot\psi\left(\alpha,w,x_j\right),\lambda\cdot\psi^2\left(\alpha,w,x_j\right)\right)\mathrm{d}z\right)\right.\\ &\qquad\left.\times f\left(w-x_j;\mu\left(x_j\right)\cdot\delta,\lambda\cdot\delta^2\right)\mathrm{d}w\right)\cdot b\left(x_j\right)\mathrm{d}x_j, \end{aligned}\tag{63}$$

- $E_{i,r,k+1}$ denotes the expected length of the system inactivity when $(k+1)$ inspections are performed over $\left[E_j^+, E_{j+1}^+\right]$

$$\begin{aligned} E_{i,r,k+1} = a\cdot&\int_0^{x_\sigma(0)}\left(\int_{x_\sigma(0)}^{x_\tau(0)}\left(\int_z^{x_\tau(0)}\rho_1(0,s)\cdot f\left(s-z;\mu(0)\cdot\psi(\alpha,z,0),\lambda\cdot\psi^2(\alpha,z,0)\right)\mathrm{d}s\right)\right.\\ &\left.\times f\left(z-w;\mu(0)\cdot\delta,\lambda\cdot\delta^2\right)\mathrm{d}v\right)\cdot f\left(w;\mu(0)\cdot k\delta,\lambda\cdot(k\delta)^2\right)\mathrm{d}z\\ &+(1-a)\cdot\int_0^{x_\sigma(x_j)}\left(\int_{x_j}^{x_\sigma(x_j)}\left(\int_{x_\sigma(x_j)}^{x_\tau(x_j)}\left(\int_z^{x_\tau(x_j)}\rho_1\left(x_j,s\right)\cdot f\left(s-z;\mu\left(x_j\right)\cdot\psi\left(\alpha,z,x_j\right),\lambda\cdot\psi^2\left(\alpha,z,x_j\right)\right)\mathrm{d}s\right)\right.\right.\\ &\left.\left.\times f\left(z-w;\mu\left(x_j\right)\cdot\delta,\lambda\cdot\delta^2\right)\mathrm{d}v\right)\cdot f\left(w-x_j;\mu\left(x_j\right)\cdot k\delta,\lambda\cdot(k\delta)^2\right)\mathrm{d}z\right)\cdot b\left(x_j\right)\mathrm{d}x_j, \end{aligned}\tag{64}$$

with a and $b\left(x_j\right)$ given from (30).

4.1.5. Expected length of the system unavailability duration over $\left[E_j^+, E_{j+1}^+\right]$

Once the system fails, it is unavailable from the failure time to starting time of the next CR and then during the CR duration. This is why an analysis of scenarios of system failure and CR allows to determine the unavailability duration of the system. In fact, the duration of system unavailability over $\left[E_j^+, E_{j+1}^+\right]$ is either

(*i*) $\int_0^{\psi\left(\alpha, X_{E_j^+}, X_{E_j^+}\right)} 1_{\left\{x_\sigma\left(X_{E_j^+}\right)\le X_{E_j^+}<L\le X_t\right\}}\mathrm{d}t + \rho_0\cdot 1_{\left\{x_\sigma\left(X_{E_j^+}\right)\le X_{E_j^+}<L\le X_{S_0}\right\}}$ if the system fails before the first inspection and a CR is performed without inspection,

(*ii*) $\int_0^{\delta} 1_{\left\{0\le X_{E_j^+}<x_\sigma\left(X_{E_j^+}\right)<L\le X_t\right\}}\mathrm{d}t+\rho_0\cdot 1_{\left\{0\le X_{E_j^+}<x_\sigma\left(X_{E_j^+}\right)<L\le X_{T_{j,1}}\right\}}$ if the system failure is detected at the 1-st inspection at which a CR starts,

(*iii*) $\int_\delta^{\delta+\psi\left(\alpha, X_{T_{j,1}}, X_{E_j^+}\right)} 1_{\left\{0\le X_{E_j^+}<x_\sigma\left(X_{E_j^+}\right)<X_{T_{j,1}}<L\le X_t\right\}}\mathrm{d}t+\rho_0\cdot 1_{\left\{0\le X_{E_j^+}<x_\sigma\left(X_{E_j^+}\right)<X_{T_{j,1}}<L\le X_{S_0}\right\}}$ if the system failure is detected after the 1-st inspection at which a CR starts,

(*iv*) $\int_{k\delta}^{(k+1)\delta} 1_{\left\{0\le X_{E_j^+}<X_{T_{j,k}}<x_\sigma\left(X_{E_j^+}\right)<L\le X_t\right\}}\mathrm{d}t+\rho_0\cdot 1_{\left\{0\le X_{E_j^+}<X_{T_{j,k}}<x_\sigma\left(X_{E_j^+}\right)<L\le X_{T_{j,k+1}}\right\}}$ if the system failure is detected at the $(k+1)$-st inspection with $k\in\mathbb{N}^*$ at which a CR starts,

(*v*) $\int_{(k+1)\delta}^{(k+1)\delta+\psi\left(\alpha, X_{T_{j,k+1}}, X_{E_j^+}\right)} 1_{\left\{0\le X_{E_j^+}<X_{T_{j,k}}<x_\sigma\left(X_{E_j^+}\right)<X_{T_{j,k+1}}<L\le X_t\right\}}\mathrm{d}t+\rho_0\cdot 1_{\left\{0\le X_{E_j^+}<X_{T_{j,k}}<x_\sigma\left(X_{E_j^+}\right)<X_{T_{j,k+1}}<L\le X_{S_0}\right\}}$ if the system failure is detected after the $(k+1)$-st inspection with $k\in\mathbb{N}^*$ at which a CR starts.

As a result, we can express the expected length of the system unavailability duration over $\left[E_j^+, E_{j+1}^+\right]$ with respect to the stationary measure π as

$$\begin{aligned} E_\pi\left[U\left(\left[E_j^+, E_{j+1}^+\right]\right)\right] = \rho_0\cdot&\left(P_{c,0}+P_{c,1,i}+P_{c,1,s}+\sum_{k=1}^{\infty}P_{c,k+1,i}+\sum_{k=1}^{\infty}P_{c,k+1,s}\right)\\ &+E_{u,0}+E_{u,1,i}+E_{u,1,s}+\sum_{k=1}^{\infty}E_{u,k+1,i}+\sum_{k=1}^{\infty}E_{u,k+1,s}, \end{aligned}\tag{65}$$

where $P_{c,0}$, $P_{c,1,i}$, $P_{c,1,s}$, $P_{c,k+1,i}$ and $P_{c,k+1,s}$ are given from (57), (58), (59), (60) and (61) respectively, and $E_{u,0}$, $E_{u,1,i}$, $E_{u,1,s}$, $E_{u,k+1,i}$ and $E_{u,k+1,s}$ are represented as follows.

- $E_{u,0}$ denotes the expectation of the duration from the system failure time to the starting time of the next CR when no inspection has been performed

$$E_{u,0} = (1-a) \cdot \int_{x_\sigma(x_j)}^{L} \left(\int_0^{\psi(\alpha,x_j,x_j)} \bar{F}\left(L - x_j; \mu\left(x_j\right) \cdot t, \lambda \cdot t^2\right) \mathrm{d}t \right) \cdot b\left(x_j\right) \mathrm{d}x_j, \tag{66}$$

- $E_{u,1,i}$ denotes the expectation of the duration from the system failure time to the 1-st inspection time at which the CR starts

$$E_{u,1,i} = a \cdot \int_0^{\delta} \bar{F}\left(L; \mu(0) \cdot t, \lambda \cdot t^2\right) \mathrm{d}t + (1-a) \cdot \int_0^{\delta} \left(\int_0^{x_\sigma(x_j)} \bar{F}\left(L - x_j; \mu\left(x_j\right) \cdot t, \lambda \cdot t^2\right) \cdot b\left(x_j\right) \mathrm{d}x_j \right) \mathrm{d}t, \tag{67}$$

- $E_{u,1,s}$ denotes the expectation of the duration from the system failure time to the starting time of the next CR when one inspection has been done

$$\begin{aligned} E_{u,1,s} = {} & a \cdot \int_{x_\sigma(0)}^{L} \left(\int_0^{\psi(\alpha,w,0)} \bar{F}\left(L - w; \mu(0) \cdot t, \lambda \cdot t^2\right) \mathrm{d}t \right) \cdot f\left(w; \mu(0) \cdot \delta, \lambda \cdot \delta^2\right) \mathrm{d}w \\ & + (1-a) \cdot \int_0^{x_\sigma(x_j)} b\left(x_j\right) \cdot \left(\int_{x_\sigma(x_j)}^{L} \left(\int_0^{\psi(\alpha,w,x_j)} \bar{F}\left(L - w; \mu\left(x_j\right) \cdot \theta, \lambda \cdot \theta^2\right) \mathrm{d}\theta \right) \cdot f\left(w - x_j; \mu\left(x_j\right) \cdot \delta, \lambda \cdot \delta^2\right) \mathrm{d}w \right) \mathrm{d}x_j, \end{aligned} \tag{68}$$

- $E_{u,k+1,i}$ denotes the expectation of the duration from the system failure time to the $(k+1)$-st inspection time at which the CR starts

$$\begin{aligned} E_{u,k+1,i} = {} & a \cdot \int_0^{\delta} \left(\int_0^{x_\sigma(0)} \bar{F}\left(L - w; \mu(0) \cdot t, \lambda \cdot t^2\right) \cdot f\left(w; \mu(0) \cdot k\delta, \lambda \cdot (k\delta)^2\right) \mathrm{d}w \right) \mathrm{d}t \\ & + (1-a) \cdot \int_0^{\delta} \left(\int_0^{x_\sigma(x_j)} \left(\int_{x_j}^{x_\sigma(x_j)} \bar{F}\left(L - w; \mu\left(x_j\right) \cdot t, \lambda \cdot t^2\right) \cdot f\left(w - x_j; \mu\left(x_j\right) \cdot k\delta, \lambda \cdot (k\delta)^2\right) \mathrm{d}w \right) \cdot b\left(x_j\right) \mathrm{d}x_j \right) \mathrm{d}t, \end{aligned} \tag{69}$$

- $E_{u,k+1,s}$ denotes the expectation of the duration from the system failure time to the starting time of the next CR when $(k+1)$ inspections have been done

$$\begin{aligned} E_{u,k+1,s} = {} & a \cdot \int_0^{x_\sigma(0)} \left(\int_{x_\sigma(0)}^{L} \left(\int_0^{\psi(\alpha,z,0)} \bar{F}\left(L - z; \mu(0) \cdot t, \lambda \cdot t^2\right) \mathrm{d}t \right) \cdot f\left(z - w; \mu(0) \cdot \delta, \lambda \cdot \delta^2\right) \mathrm{d}z \right) \\ & \times f\left(w; \mu(\grave{a}) \cdot k\delta, \lambda \cdot (k\delta)^2\right) \mathrm{d}w + (1-a) \cdot \int_0^{x_\sigma(x_j)} \left(\int_{x_j}^{x_\sigma(x_j)} \left(\int_{x_\sigma(x_j)}^{L} \left(\int_0^{\psi(\alpha,z,x_j)} \bar{F}\left(L - z; \mu\left(x_j\right) \cdot t, \lambda \cdot t^2\right) \mathrm{d}t \right) \right.\right. \\ & \left.\left. \times f\left(z - w; \mu\left(x_j\right) \cdot \delta, \lambda \cdot \delta^2\right) \mathrm{d}z \right) \cdot f\left(w - x_j; \mu\left(x_j\right) \cdot k\delta, \lambda \cdot (k\delta)^2\right) \mathrm{d}w \right) \cdot b\left(x_j\right) \mathrm{d}x_j, \end{aligned} \tag{70}$$

with a and $b\left(x_j\right)$ given from (30).

4.2. Maintenance policy optimization

Optimizing the $(\delta, \sigma, \alpha, \tau)$ policy returns to seek the optimal configuration $\left(\delta_{opt}, \sigma_{opt}, \alpha_{opt}, \tau_{opt}\right)$ that minimizes the long-run maintenance cost rate $C_\infty(\delta, \sigma, \alpha, \tau)$

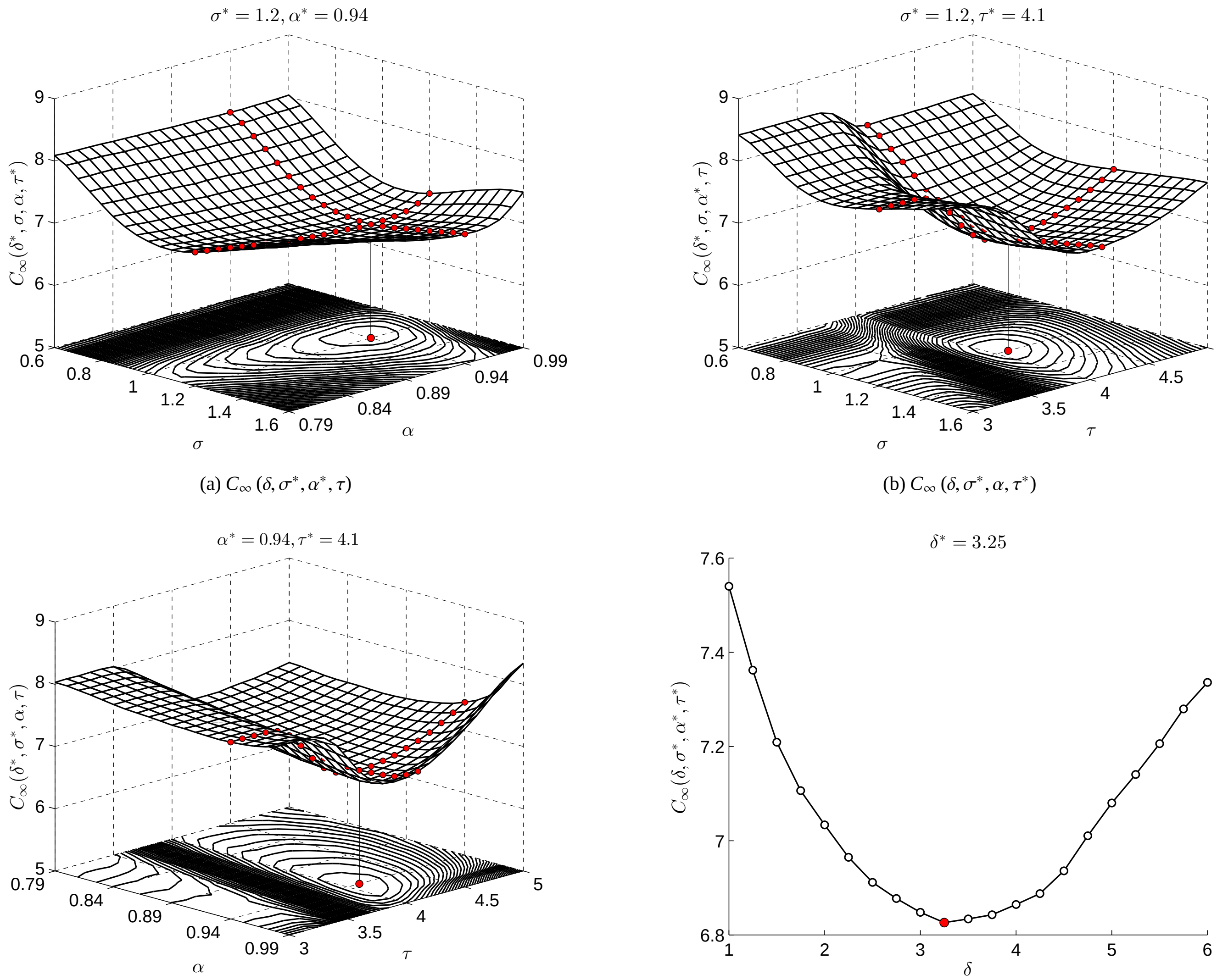
$$\left(\delta_{opt}, \sigma_{opt}, \alpha_{opt}, \tau_{opt}\right) = \arg \min_{(\delta,\sigma,\alpha,\tau)} \left\{ C_\infty(\delta, \sigma, \alpha, \tau) : \delta > 0, \sigma > 0, 0 < \alpha < 1, \tau \geq \rho_0 \right\}, \tag{71}$$

where ρ_0 denotes the required duration for a replacement. Even though the closed-form expression of $C_\infty(\delta, \sigma, \alpha, \tau)$ is available, its complexity does not allow an analytical solution for (71), and numerical methods should be used instead.

To prove the existence of $\left(\delta_{opt}, \sigma_{opt}, \alpha_{opt}, \tau_{opt}\right)$ numerically, we vary δ, σ, α and τ in a wide rank, and observe the form of $C_\infty(\delta, \sigma, \alpha, \tau)$. Repeating such an experiment for various configurations of maintenance costs (i.e., C_m, C_r, C_o, C_i and C_u) and of system characteristics (i.e., L, μ_0, $\mu_1(\cdot)$, λ, ρ_0, $\rho_1(\cdot,\cdot)$, $g(\cdot \mid \cdot,\cdot)$), we can draw a conclusion about the existence of $\left(\delta_{opt}, \sigma_{opt}, \alpha_{opt}, \tau_{opt}\right)$. Although this numerical approach cannot cover all possible configurations of the maintained system, it is still an alternative solution when analytical approach is impossible. To find the optimum $\left(\delta_{opt}, \sigma_{opt}, \alpha_{opt}, \tau_{opt}\right)$, we propose using the generalized pattern search algorithm [58]. Numerous numerical experiments confirm that this algorithm allows to speed up the optimum finding regardless of chosen initial values.

4.3. Numerical illustration

As an illustration, we apply the maintenance costs $C_m = 2$, $C_r = 20$, $C_o = 100$, $C_i = 5$ and $C_u = 15$ to the system considered in Subsection 3.3. We compute and sketch in Figure 6 the long-run maintenance cost rate $C_\infty(\delta, \sigma, \alpha, \tau)$ by varying δ from 1 to 6 with step 0.25, σ from 0.6 to 1.6 with step 0.05, α from 0.79 to 0.99 with step 0.01, and τ from 3 to 5 with step 0.1. In the Figures 6a, 6b



(a) $C_\infty(\delta, \sigma^*, \alpha^*, \tau)$

(b) $C_\infty(\delta, \sigma^*, \alpha, \tau^*)$

(c) $C_\infty(\delta, \sigma, \alpha^*, \tau^*)$

(d) $C_\infty(\delta, \sigma^*, \alpha^*, \tau^*)$

Figure 6: Shapes of $C_\infty(\delta, \sigma, \alpha, \tau)$ and $C_\infty(\delta^*, \sigma^*, \alpha^*, \tau^*) = 6.8163$

and 6c, 2 among 4 decision variables are fixed and 2 others vary, while in Figure 6d, only δ varies. The convex forms of $C_\infty(\delta, \sigma, \alpha, \tau)$ confirm the existence of optimal configuration for the $(\delta, \sigma, \alpha, \tau)$ policy.

In Figure 6, we find the minimum value $C_\infty(\delta^*, \sigma^*, \alpha^*, \tau^*) = 6.8163$ at $\delta^* = 4.1$, $\sigma^* = 1.2$, $\alpha^* = 0.94$, $\tau^* = 4.1$. However, this configuration is not optimal yet, because the chosen lattices for δ, σ, α and τ are not fine enough. To seek the "real" optimum $\left(\delta_{opt}, \sigma_{opt}, \alpha_{opt}, \tau_{opt}\right)$, we use the `patternsearch` solver of Matlab's Global Optimization Toolbox. As shown in Figure 7, the optimal configuration of the above maintained system is reached at $\delta_{opt} = 3.375$, $\sigma_{opt} = 1.1563$, $\alpha_{opt} = 0.94688$ and $\tau_{opt} = 4$ with $C_\infty\left(\delta_{opt}, \sigma_{opt}, \alpha_{opt}, \tau_{opt}\right) = 6.8085$.

5. Numerical assessment

To assess the economic performances of the $(\delta, \sigma, \alpha, \tau)$ policy, comparative studies with benchmark policies are proposed in this section. The studies take into account the impacts of different maintenance costs and various system deterioration characteristics. Notwithstanding, we just present hereinafter the results for the maintained system defined by

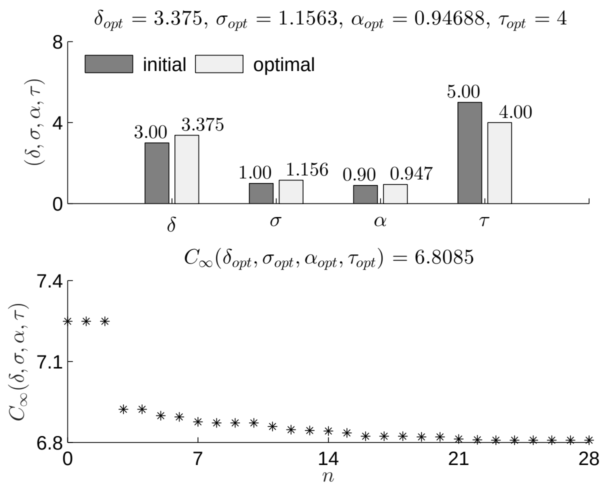


Figure 7: Optimization of $(\delta, \sigma, \alpha, \tau)$ policy with Matlab's `patternsearch` solver

- a failure threshold: $L = 15$,
- a degradation process with linear shape function: $\mathcal{IGP}\left(\mu\left(x_j\right), \lambda\right) = \mathcal{IGP}\left(\mu_0 + \mu_1 \cdot x_j, \lambda\right) = \mathcal{IGP}\left(1 + \mu_1 \cdot x_j, 4\right)$,
- a linear IR duration: $\rho\left(X_{E_j^+}, X_{S_j}\right) = \rho_0 + \rho_1\left(X_{E_j^+}, X_{S_j}\right) = \rho_0 + \rho_{1,1} \cdot X_{E_j^+} + \rho_{1,2} \cdot X_{S_j} = 1 + 0.1 \cdot X_{E_j^+} + 0.2 \cdot X_{S_j}$,
- a continuous uniform pdf for $g\left(x_{j+1} \mid X_{E_j^+}, X_{S_j}\right)$.

The linear form of $\mu\left(x_j\right)$ and $\rho_1\left(X_{E_j^+}, X_{S_j}\right)$ is inspired by [18] and [59] respectively, while the other parameters are arbitrarily chosen. The applied maintenance costs are fixed at $C_m = 2$, $C_o = 100$, $C_i = 5$ and $C_u = 15$. Some missing values (i.e., μ_1 and C_r) will be stated latter depending on concrete numerical illustrations.

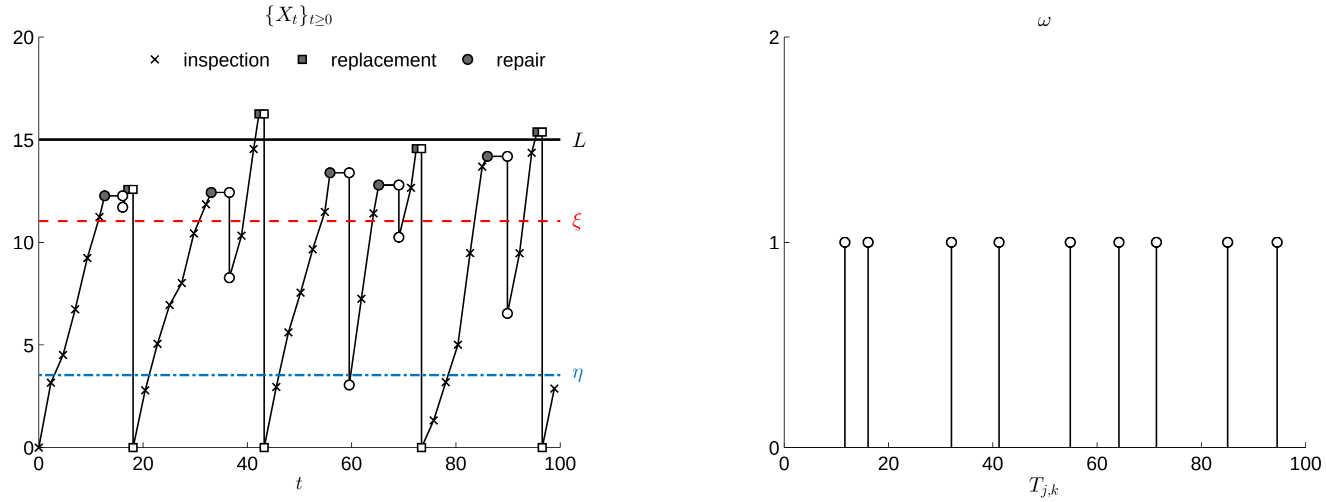
5.1. Benchmark maintenance policies

Three PdM policies are used as benchmarks. The first one, called $(\delta, \zeta, \omega, \eta)$, is static in the sense that repair and replacement decisions are not adaptive to the system degradation behavior. Meanwhile, as extreme cases of the $(\delta, \sigma, \alpha, \tau)$ policy, the two others, called $(\delta, \sigma, \alpha, \tau \to \rho_0)$ and $(\delta, \sigma, \alpha, \tau \to +\infty)$, are adaptive, but either the IR or perfect replacement is used as preventive actions.

5.1.1. $(\delta, \zeta, \omega, \eta)$ policy

The $(\delta, \zeta, \omega, \eta)$ policy generalizes the PdM policies proposed in [48] and [53]. Over the cycle $\left[E_j^+, E_{j+1}^+\right]$, the system is regularly inspected at times $T_{j,k} = E_j + k \cdot \delta$, with $k = 0, 1, 2, \ldots$. to reveal its degradation level. Given $X_{T_{j,k}}$, we adopt the following decision rules.

1. If $X_{T_{j,k}} \geq L$, the failed system is correctively replaced immediately at $T_{j,k}$. After the CR with duration ρ_0, the system is AGAN (i.e., $X_{E_{j+1}^+} = 0$ with $E_{j+1} = T_{j,k} + \rho_0$).
2. If $\xi \leq X_{T_{j,k}} < L$, the system is still running at $T_{j,k}$ and its RUL can be predicted with an acceptable precision (see Subsection 2.2). So, no further inspection is needed, and the next maintenance is planned $\omega \geq 0$ time units later (i.e., at $S_j = T_{j,k} + \omega$). The nature of the maintenance action depends on both the degradation levels X_{S_j} and $X_{E_j^+}$.
   (a) If $X_{S_j} \geq L$, a CR with duration ρ_0 is triggered immediately at S_j to reset the failed system to an AGAN state (i.e., $X_{E_{j+1}^+} = 0$ with $E_{j+1} = S_j + \rho_0$).
   (b) If $X_{S_j} < L$ and $X_{E_j^+} \geq \eta$, the system is still running at S_j, and the "high" value of $X_{E_j^+}$ implies that an IR is no longer suitable for the current maintenance. So, a PR should be carried out at S_j instead. After the PR, the system is AGAN (i.e., $X_{E_{j+1}^+} = 0$ with $E_{j+1} = S_j + \rho_0$).
   (c) If $X_{S_j} < L$ and $X_{E_j^+} < \eta$, a preventive IR is immediately performed at S_j on the running system. It takes $\rho\left(X_{E_j^+}, X_{S_j}\right)$ time units, and returns the system degradation $X_{E_{j+1}^+}$ to a random level between $X_{E_j^+}$ and X_{S_j} such that $X_{E_{j+1}^+} \sim g\left(y \mid X_{E_j^+}, X_{S_j}\right)$, where $E_{j+1} = S_j + \rho\left(X_{E_j^+}, X_{S_j}\right)$ and g is a known truncated pdf.
3. If $X_{T_{j,k}} < \xi$, additional inspections are required to reinforce the precision of RUL prediction. Accordingly, the decisions is postponed to the next inspection time at $T_{j,k+1} = T_{j,k} + \delta$.



(a) System degradation state and maintenance actions

(b) Waiting time before the beginning of maintenance actions

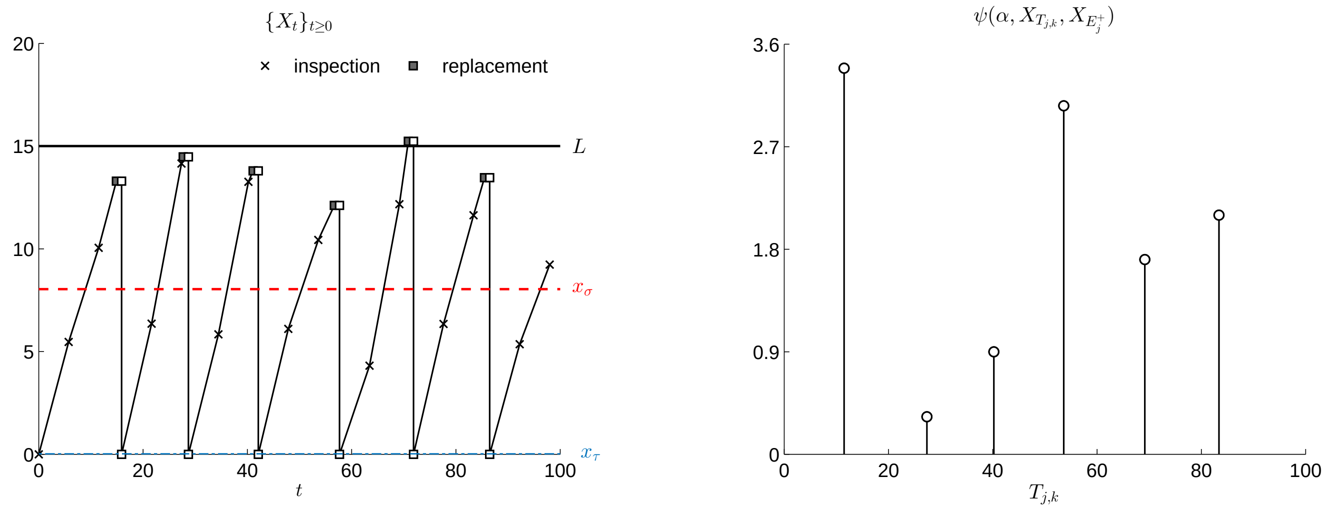
Figure 8: Schematic behavior of the maintained system with $\delta = 2.33$, $\xi = 11.03$, $\omega = 1$ and $\eta = 3.53$ (i.e., optimal $(\delta, \zeta, \omega, \eta)$ policy when $\mu_1 = 0.1$ and $C_r = 20$)

The next maintenance cycle begins at E_{j+1}^{+} with initial deterioration level $X_{E_{j+1}^{+}}$. The inspection period δ, the degradation thresholds ξ and η, and the waiting time ω are decision variables. Its long-run maintenance cost model is developed in the same way as the $(\delta, \sigma, \alpha, \tau)$ policy. Figure 8 illustrates the schematic behavior of the maintained system under the $(\delta, \zeta, \omega, \eta)$ policy. Its meaning is very similar to the one of Figure 3. Clearly, with fixed ξ, ω and η, the maintenance decisions of the $(\delta, \zeta, \omega, \eta)$ policy cannot adapt to the system degradation behavior. So, the comparison between the $(\delta, \zeta, \omega, \eta)$ and $(\delta, \sigma, \alpha, \tau)$ policies allows to see the added values of adaptive maintenance decisions.

5.1.2. $(\delta, \sigma, \alpha, \tau \to \rho_0)$ *policy and* $(\delta, \sigma, \alpha, \tau \to +\infty)$ *policy*

These two benchmarks policies are extreme cases of the $(\delta, \sigma, \alpha, \tau)$ policy, where either replacement or repair is used as preventive maintenance action.

1. When $\tau \to \rho_0$, only replacement is implemented for preventive action because $P\left(\rho\left(X_{E_j^+}, X_{S_j}\right) \geq \tau\right) \to 1$. The $(\delta, \sigma, \alpha, \tau)$ policy becomes a pure PR policy $(\delta, \sigma, \alpha, \tau \to \rho_0)$ (see also [53]). The schematic behavior of the maintained system under the $(\delta, \sigma, \alpha, \tau \to \rho_0)$ policy is sketched in Figure 9. We find that the waiting time $\psi(\cdot)$ well adapts to the system degradation.



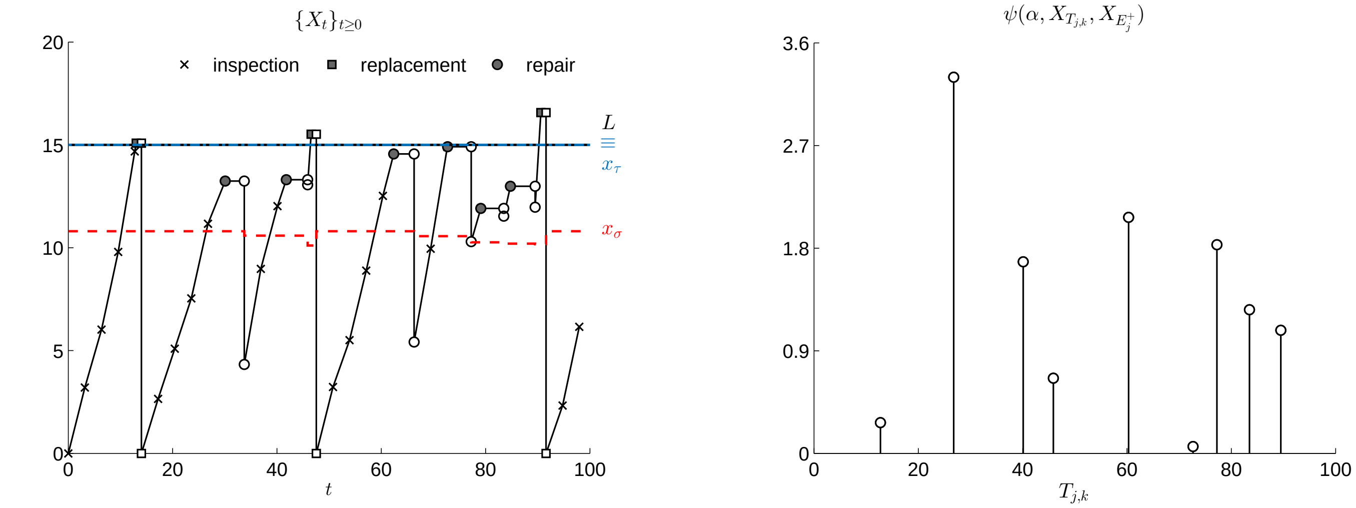
(a) System degradation state and maintenance actions

(b) Waiting time before the beginning of maintenance actions

Figure 9: Schematic behavior of the maintained system with $\delta = 5.75$, $\sigma = 1.3$, $\alpha = 0.939$ (i.e., optimal $(\delta, \sigma, \alpha, \tau \to \rho_0)$ policy when $\mu_1 = 0.1$ and $C_r = 20$)

2. When $\tau \to +\infty$, only repair is carried out in preventive decision because $P\left(\rho\left(X_{E_j^+}, X_{S_j}\right) \geq \tau\right) \to 0$. So, $(\delta, \sigma, \alpha, \tau)$ policy returns to a pure preventive IR policy $(\delta, \sigma, \alpha, \tau \to +\infty)$. The schematic behavior of the associated maintained system is illustrated in Figure 10. Obviously, the flexibility of the $(\delta, \sigma, \alpha, \tau \to +\infty)$ policy is reflected not only by the dynamic waiting time $\psi(\cdot)$, but also by the varied value of x_σ.

The comparison between the $(\delta, \sigma, \alpha, \tau)$ policy and its extreme cases will justify the effectiveness of hybrid maintenance decisions.



(a) System degradation state and maintenance actions

(b) Waiting time before the beginning of maintenance actions

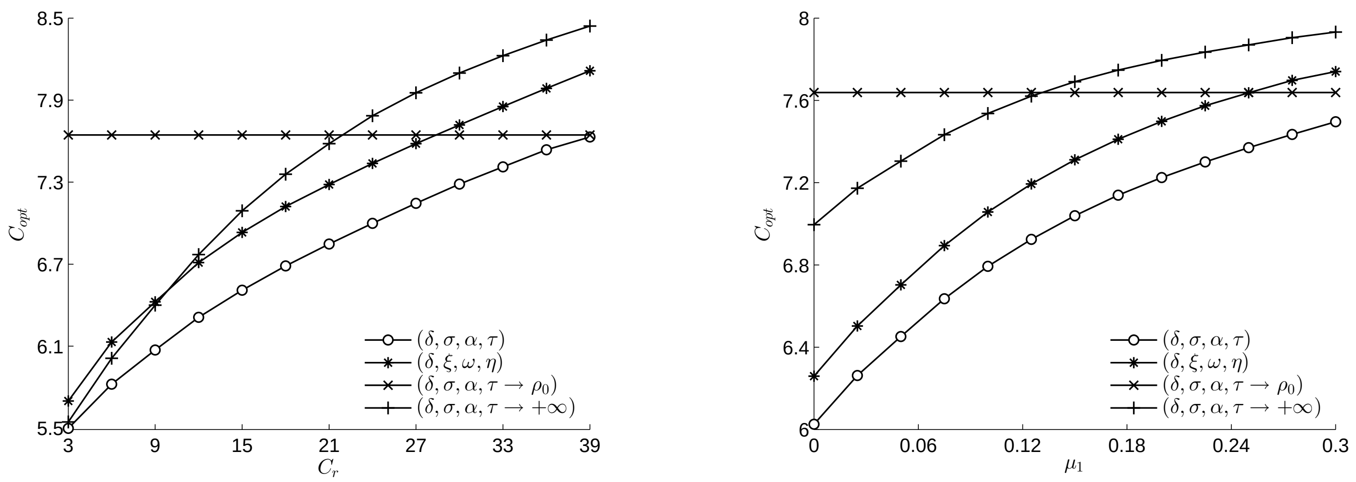
Figure 10: Schematic behavior of the maintained system with $\delta = 5.75$, $\sigma = 1.3$, $\alpha = 0.939$ (i.e., optimal $(\delta, \sigma, \alpha, \tau \to +\infty)$ policy when $\mu_1 = 0.1$ and $C_r = 20$)

5.2. *Case studies and comparison results*

To understand the impacts of the maintenance costs and the system characteristic on the cost saving, we consider 2 following case studies:

1. *sensitivity to repair cost*: C_r varies from 3 to 39 with step 3, and $\mu_1 = 0.1$,
2. *sensitivity to degradation rate*: $C_r = 20$, and μ_1 varies from 0 to 0.3 with step 0.03.

The associated optimal long-run maintenance cost rate of the 4 considered maintenance policies are shown in Figures 11a and 11b. In each figure, lowers curve correspond to higher economic performances.



(a) Sensitivity to C_r

(b) Sensitivity to μ_1

Figure 11: Evolution of the optimal long-run maintenance cost rates

The $(\delta, \sigma, \alpha, \tau \to \rho_0)$ policy has a constant optimal cost rate in both the situations, because this pure PR policy is independent of C_r and λ_1. Meanwhile, using IR as a preventive maintenance action, the other PdM policies have evidently increasing cost rate with respect to C_r and λ_1. Comparing the cost curves of the $(\delta, \sigma, \alpha, \tau \to \rho_0)$ policy and $(\delta, \sigma, \alpha, \tau \to +\infty)$ policy, we find out the effectiveness of each kind of maintenance actions. The replacement is better if C_r or μ_1 is small, otherwise the repair is more profitable. To make use of their advantage, hybrid maintenance decisions should be resorted to. Indeed, as clearly shown in Figures 11a and 11b, the $(\delta, \sigma, \alpha, \tau)$ policy always saves more maintenance cost, and just returns to the pure policies in worse cases. Now, looking at the cost curves of the $(\delta, \sigma, \alpha, \tau)$ policy and the $(\delta, \zeta, \omega, \eta)$ policy, the former always gives more cost saving. The profit is even higher when C_r or μ_1 increases. This implies that adaptive decisions allows to resist better to the negative impact of maintenance costs, and to well adapt to the variation of the system degradation behavior.

6. Conclusion and perspectives

The focus of this paper is to develop a cost-effective PdM model for deteriorating systems using periodic inspection, IR and perfect replacement. The development consists of four steps: continuous degradation modeling, maintenance effect modeling, adaptive PdM policy elaboration, and long-run maintenance cost rate evaluation. The connection between these steps is especially highlighted by (*i*) the consideration of the past dependency of IR actions in the IG degradation process, (*ii*) the use of estimated system RUL and maintenance duration to enable adaptive PdM decisions, and (*iii*) the probabilistic study of the behavior of maintained system at steady state based on the semi-regenerative theory. Various numerical experiments and comparative studies show that the developed adaptive PdM model is more flexible, and hence more profitable than related PdM models.

Despite very encouraging results, our PdM model is still based on a strong assumption that the model parameters are already known. However, these parameters are usually unknown in practice, and should be estimated from the available degradation and maintenance data. This is why our future works will focus on overcoming this drawback. Some potential perspectives are as follows. Firstly, we think of building a testing platform able to deliver both the degradation and maintenance data. The reason is that most existing benchmark data-sets[1] are interesting to test prognostic algorithms, but not suitable for the PdM modeling. Once the required data are available, we carry out the distribution selection for the past-dependent IR and the parameters estimation for the IG degradation process. We believe that statistical methods proposed in [18, 60, 61] can help. Finally, we shall adapt our PdM model to online application by updating the decision variables following available monitoring data. Such an online model is currently studied in [33] under Bayesian framework, but the considered PdM policy remains relatively simple.

Acknowledgment

This work was partially supported by the Fond Européen de Développement Régional (FEDER) under the framework of Project MAP 4.0.

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