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Abstract

Imperfect maintenance models are widely used in reliability engineering. This paper discusses relevant asymptotic properties for the steady-state virtual age processes. It is shown that the limiting distributions of age, the residual lifetime and the spread that describe an ordinary renewal process can be generalized to the stable virtual age process, although the cycles of the latter are not independent. Asymptotic distributions of the virtual age at time t, as well as of the virtual ages at the start and the end of a cycle containing t (as t tends to infinity) are explicitly derived for two popular in practice imperfect maintenance models, namely, the Arithmetic Reduction of Age (ARA) and the Brown-Proschan (BP) models. Some applications of the obtained results to maintenance optimization are discussed.

Keywords: Maintenance, Imperfect repair, Virtual age process, Renewal theory, Limiting distribution
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1. Introduction

It is well known that the origin of the renewal theory (Feller (1968)) is in numerous industrial applications. For instance, the renewal function in practice can be interpreted as the mean number of replacements/perfect repairs for a system operating in a given interval of time. Thus, the mean number of the required spare parts can be estimated and the probability of the spare parts shortage as well (the latter is usually performed only for some specific cases, e.g., for the Homogeneous Poisson Process of renewals). Furthermore, when describing the performance of, for example, repairable production systems, renewal processes are generalized to the corresponding renewal reward processes that allow (among other things) for obtaining the optimal, long-run maintenance policies (Cox & Isham (1980)).

As the cycles of an ordinary (standard) renewal process are i.i.d. random variables, they naturally model the process of perfect repairs (with a negligible repair time). Thus, the perfect, or the As-Good-As-New (AGAN) repair, restores a system to the state of a new system, whereas the minimal, or the As-Bad-As-Old (ABAO) repair, restores it to the state it had just prior to failure. It is well known that the latter type of repair processes can be described by the non-homogeneous Poisson Process (NHPP) (Ascher & Feingold (1984)). In reality, however, the efficiency of a repair action is often between the AGAN and the ABAO repair, as a system can be effectively repaired without necessarily being totally renewed. This action is usually referred to as an imperfect maintenance/repair (Pham & Wang (1996)) and some basic modeling approaches are reviewed in Lindqvist (2006). The maintenance optimization problems under the imperfect repair has been studied from different perspectives: periodic preventive maintenance(PM) has been addressed in Zequeira & Brenguer (2006); age-based PM was discussed in Huynh et al. (2012) and El-Ferik (2008); Gilardoni et al. (2016) showed that when the failure history is available, the failure-limit PM policy could be more cost-efficient than the age-based PM; Yang et al. (2019) and Wu et al. (2019) combined the imperfect repair with delayed time concept; Mullor et al. (2019) considered multiple failure modes. Applications to the data-driven modeling of repair processes can be found in, e.g., Baker (2001) and Dijoux & Gaudoin (2013),Wang & Pham (2006).

Virtual Age (VA) models (Kijima et al. (1988)) are the most common in reliability practice imperfect maintenance models when the age of a repairable system is assumed to depend not only on the time elapsed since the last repair (as in the case of the perfect repair), but on some virtual age between zero (perfect repair) and the corresponding calendar age (minimal repair). A virtual age model is fully defined by the age reduction mechanism and by the Cdf of the time to failure (or by the corresponding failure rate) of a new system that is also called the baseline or initial Cdf (failure rate). Two popular virtual age assumptions have been proposed in Kijima (1989), where the repair efficiency is characterized by a random variable ρ, contained in the closed interval [0,1]. Kijima's Type I model assumes that the reduced amount of virtual age after a repair is proportional to the last inter-failure time, whereas in Kijima's Type II model, the reduction of the virtual age after a repair is proportional to that just before the repair. These two models were generalized in Doyen & Gaudoin (2004) by introducing the model of Arithmetic Reduction of Age of memory m, denoted as ARA_m. The values $m = 1$ and $m = \infty$ correspond to Kijima's Type I and Kijima's Type II models, respectively, with constant reduction factors. Theoretical results on these models can be found in Malik (1979), Kijima & Sumita (1986), Yevkin (2011), Kijima (1989), Doyen & Gaudoin (2004) and Nguyen et al. (2017). The maintenance scheduling and optimization based on these results have been reported in Dimitrakos & Kyriakidis (2007), Jiang et al. (2001), Kijima et al. (1988), Love et al. (2000) and Makis & Jardine (1993).

Another widely-used imperfect maintenance model is the Brown-Proschan (BP) model (Brown & Proschan (1983)). The repair action in this model can be either AGAN with probability p or ABAO with probability $1 - p$. This model is relevant, e.g., in situations where some of the minor failures of a complex system are minimally repaired, whereas other, more serious failures result in replacement of the failed system. Similar to the reduction factor in Kijima's-type models, parameter p in the BP model determines the age reduction mechanism. Theoretical results on the BP model and the corresponding statistical inference issues have been addressed in Whitaker & Samaniego (1989), Hollander et al. (1992), Lim (1998) and Laurent (2011), whereas its industrial applications have been studied in Block et al. (1985), Kumar & Klefsj (1992), Langseth & Lindqvist (2003) and Finkelstein & Shafiee (2017), to name a few.

Ordinary renewal processes are stationary in the sense that the corresponding renewal density function tends to a constant as time tends to infinity. The NHPP that describes minimal repairs is, obviously, non-stationary and if, e.g., its rate is increasing, the failures are arriving more frequently with time. Kijima Type I and Geometric Process (Yeh (1988)), like NHPP, are non-stationary and can be used to model the lifetime with trends. Although there are many publications on various applications of the virtual age models in reliability, not much has been done so far in the literature on description of the relevant asymptotic properties for the corresponding virtual age processes. For instance, Finkelstein (2007) and Finkelstein (2008) have proven that under certain assumptions, there exist some limiting distributions of the age at the start and the end of each cycle. Furthermore, in Laurent (2011) the limiting distributions of the virtual ages and of the cycle durations were obtained analytically for the specific case of the Brown-Proschan (BP) repair process, whereas similar results have been reported with respect to the ARA_∞ process with Weibull baseline distribution in Nguyen et al. (2017).

It should be noted that the limiting properties of the ordinary renewal processes are especially important in various applications. For instance, obtaining the corresponding renewal functions can be computationally difficult and simple asymptotic values provided by the renewal-type theorems are very effective in practice. Another example is the alternating renewal process. The stationary availability in this case, which is usually of the main interest, is obtained in a simple way via the mean up and down times of a system. The life cycles of many industrial systems are quite long meaning that a large number of maintenance actions are performed. Moreover, at many instances, the operational data is recorded only when a system enters its stable regime. Therefore, the importance of asymptotic methods in the described context is hard to overestimate.

The study of asymptotic properties of the imperfect repair processes that more adequately than ordinary renewal processes describe maintenance of the real-world systems, seems to be *a natural and practically sound task* that is addressed in the current paper. For achieving this goal, we had to answer first the following questions: can asymptotic results for the age, the residual lifetime and the spread for ordinary renewal processes be generalized (and under what conditions) to the case of the imperfect repair processes? What are the asymptotic distributions for these quantities? To answer these questions, certain theoretical results had to be obtained and illustrated afterwards by several practical examples.

The rest of the paper is organized as follows: in Section 2, some general properties of the relevant virtual age models are presented. Sections 3 and 4 are devoted to generalization of the limiting distributions of the age, the residual lifetime and the spread to the case of stable virtual age models. Finally, some applications in maintenance optimization are discussed in Section 5, whereas concluding remarks are given in Section 6.

2. Virtual age process

Denote by $\{T_i\}, i \in N$ the successive failure/repair times of a system with $T_0 = 0$ and $\{X_i\}$ the inter-failure times (also called intervals or cycles) with $X_0 = 0$ and $X_i = T_i - T_{i-1}$ for $i \geq 1$. The cumulative number of observed failures up to t is denoted by $N_t = \sum_{i=1}^{\infty} \mathbb{1}_{\{T_i \leq t\}}$. The failure process can be defined by its stochastic intensity:

$$\forall t \geq 0, \lambda_t = \lim_{\Delta t \to 0} \frac{1}{\Delta t} P(N_{t+\Delta t} - N_{t^-} = 1 | \mathcal{H}_{t^-}), \tag{2.1}$$

where $\mathcal{H}_{t^-}$ is the failure history.

Let the baseline/initial failure rate and the corresponding cumulative failure rate of a new system be $\lambda(t)$ and $\Lambda(t) = \int_0^t \lambda(u)du$, respectively. In practice,

$\lambda(t)$ is often given by the 2-parameter Weibull distribution, where the shape parameter β determines whether $\lambda(t)$ is increasing or decreasing.

$$\forall t \geq 0, \lambda(t) = \alpha\beta t^{\beta-1}. \tag{2.2}$$

Let a new system with a lifetime T described by the Cdf $F(t)$ start operation at $t = 0$. Then, at age x, the Cdf of the remaining lifetime is given by $F(t|x) = 1 - \overline{F}(t+x)/\overline{F}(x)$. Assume that after the instantaneous maintenance (corrective or preventive) carried out at time t, the remaining lifetime is defined as the lifetime of a new and unmaintained system having age y, where $y < t$. Then y is called the *virtual age*, the calendar age after this operation is, obviously, still t.

A repair with a negligible duration is carried out immediately after T_i, and is supposed to reduce a system's age to $A_i, A_0 = 0$, which is called the virtual age after the i-th failure. Then, the remaining lifetime of the repaired system does not depend on the entire failure/repair history, but depends on the virtual age of the system after the most recent repair. Mathematically, this is described by

$$\forall i \in N, \forall t \geq 0, P\left(X_{i+1} \leq t | T_1, T_2 \ldots T_i\right) = F(t|A_i). \tag{2.3}$$

At a given time t, the total number of failures before t is denoted by N_{t-}. Thus, $A_{N_{t-}}$ is the latest virtual age, $T_{N_{t-}}$ is the time of the last repair before t, and $T_{N_{t-}+1}$ is the time of the next repair after t. The time elapsed since the last repair before t, also called the backward recurrence time (Cox & Miller (1965)), is denoted by B_t, with $B_t = t - T_{N_{t-}}$. The remaining lifetime at t or the forward recurrence time, denoted δ_t, is the time till the next failure: $\delta_t = T_{N_{t-}+1} - t$. The spread Y_t is the total duration of a cycle that contains t: $Y_t = B_t + \delta_t$.

The virtual age at the start of the cycle containing t, denoted by V_t^s, is the virtual age after the latest repair: $V_t^s = A_{N_{t-}}$. The virtual age at time t, V_t, is obtained from the latest virtual age and the time since the last repair: $V_t = A_{N_{t-}} + t - T_{N_{t-}}$. Finally, the virtual age at the end of the cycle containing t, denoted by V_t^e, is the virtual age just before the next repair after t: $V_t^e = V_t + \delta_t$. For convenience, the above defined notations are listed in table 1.

Table 1: Notations in stable Virtual Age models

Notation	Interpretation	Expression
X_n	duration of the n-th cycle	$X_n = T_n - T_{n-1}$
A_n	virtual age after the n-th repair	
X_∞	duration of the asymptotic cycle	$X_\infty = \lim_{n\to\infty} X_n$
R_{X_∞}	limiting survival function of X_∞	$R_{X_\infty}(t) = P(X_\infty > t)$
f_{X_∞}	limiting pdf of X_∞	$f_{X_\infty}(t) = -\frac{d}{dt} R_{X_\infty}(t)$
A_∞	asymptotic virtual age at the start of a cycle	$A_\infty = \lim_{n\to\infty} A_n$
R_{A_∞}	limiting survival function of A_∞	$R_{A_\infty}(t) = P(A_\infty > t)$
f_{A_∞}	limiting pdf of A_∞	$f_{A_\infty}(t) = -\frac{d}{dt} R_{A_\infty}(t)$
μ	mean cycle duration in stationary state	$\mu = E(X_\infty) = \int_0^\infty R_{X_\infty}(x)\mathrm{d}x$
B_t	backward recurrence time at t	$B_t = t - T_{N_{t^-}}$
δ_t	remaining lifetime (forward recurrence time) at t	$\delta_t = T_{N_{t^-}+1} - t$
Y_t	spread at time t	$Y_t = B_t + \delta_t$
V_t^s	virtual age at the start of the cycle containing t	$V_t^s = A_{N_{t^-}}$
V_t	virtual age at time t	$V_t = A_{N_{t^-}} + t - T_{N_{t^-}}$
V_t^e	virtual age at the end of the cycle containing t	$V_t^e = V_t + \delta_t$

It is well known (Ross (1995)) that the backward recurrence time B_t, the remaining lifetime δ_t and the spread Y_t for ordinary renewal processes are described by some limiting (as t tends to infinity) non-dependent on t distributions. We will show that under certain assumptions, the similar results hold for some virtual age processes. Specifically, B_t, δ_t and Y_t for these processes will be also described by some limiting distributions. Based on these results, we can further derive the limiting distributions of virtual ages related to t, i.e., V_t^s, V_t and V_t^e.

In what follows in this section, two important imperfect repair models, namely, the ARA_∞ and the BP models, are briefly introduced. We are interested particularly in these models as they are intensively used in various reliability applications and because they have different age reduction mechanisms. As a result, the limiting distributions of V_t^s, V_t and V_t^e for these two processes have different forms.

2.1. ARA_∞ model

The Arithmetic Reduction of Age with infinite memory model (Doyen & Gaudoin (2004)) assumes that when a repair is carried out, the VA is reduced proportionally to that just before the maintenance:

$$A_i = (1-\rho)\left(A_{i-1} + X_i\right), i = 1, 2, \ldots, \tag{2.4}$$

where $\rho \in [0, 1]$ is called the restoration/reduction factor representing the efficiency of maintenance. The values, $\rho = 1$ and $\rho = 0$ correspond, respectively, to the perfect maintenance (AGAN) and the minimal repair (ABAO). Nguyen et al. (2017) have considered the ARA_∞ model for the Weibull baseline distribution, i.e.,

$$F(t) = 1 - e^{-\alpha t^{\beta}}, \alpha > 0, \beta > 0. \tag{2.5}$$

Thus, the corresponding VA process is fully determined by the triple (α, β, ρ). The distributions of X_n and A_n can be found in Nguyen et al. (2017). The limiting distributions as n tends to infinity, denoted R_{X_∞} and R_{A_∞}, are given below:

$$R_{A_\infty}(t) = \sum_{k=1}^{\infty} \frac{1}{(q,q)_\infty \left(\frac{1}{q}, \frac{1}{q}\right)_{k-1}} e^{-\frac{\alpha t^{\beta}}{q^k}}, \tag{2.6}$$

$$R_{X_\infty}(t) = \sum_{k=1}^{\infty} \frac{1}{q^k (q,q)_\infty \left(\frac{1}{q}, \frac{1}{q}\right)_{k-1}} \int_0^\infty \alpha\beta x^{\beta-1} e^{-\alpha(x+t)^\beta + \alpha\left(1-q^{-k}\right)x^\beta} \mathrm{d}x, \tag{2.7}$$

where $q = (1-\rho)^\beta$ and $(a,q)_k = \prod_{j=0}^{k-1}\left(1 - aq^j\right)$ is the q-Pochhammer symbol.

2.2. Brown-Proschan model

The BP model assumes that the repair after a failure is perfect with probability p and is minimal with probability $(1-p)$. Therefore, suppose that the repair effects are unknown and can be defined by the variables B_n:

$$B_n = \begin{cases} 1 & \text{if the n-th repair is AGAN} \\ 0 & \text{if the n-th repair is ABAO.} \end{cases} \tag{2.8}$$

Define A_n as the time elapsed between T_n and the last perfect repair. Thus,

$$A_n = (1 - B_n)(X_n + A_{n-1}). \tag{2.9}$$

The distribution and expectation of A_∞ and of X_∞ are given by Laurent (2011):

$$F_{A_\infty}(x) = 1 - (1-p)e^{-p\Lambda(x)}, \quad E(A_\infty) = (1-p)\int_0^\infty e^{-p\Lambda(x)} dx, \tag{2.10}$$

$$R_{X_\infty}(x) = p\int_0^\infty \lambda(x+v)e^{-\Lambda(x+v)+(1-p)\Lambda(v)} dv, \quad E(X_\infty) = p\int_0^\infty e^{-p\Lambda(x)} dx. \tag{2.11}$$

3. Asymptotic distributions of backward recurrence time, residual lifetime and spread in stable virtual age processes

Various generalizations of the 'standard renewal theory' were addressed in the literature in a number of publications. With relevance to our topic, the following papers (to name a few) can be of interest. For example, Chow & Robbins (1963) considered the renewal theory in sequences with dependent and non-identically distributed intervals; Dagpunar (1997) studied the renewal-type equations for a generalized Kijima type II process (see also Finkelstein (2007) and Finkelstein (2008)). Lam & Lehoczky (1991) considered the generalizations of renewal theory to the superposition of renewal processes.

Our interest lies in the virtual age processes with cycles X_n converging in distribution to X_∞ as n tends to infinity. This property guarantees that the cycles are asymptotically identically distributed. The following theorem defines distributions of the backward recurrence time B_t, the residual lifetime δ_t and the spread Y_t for these *stable* virtual age processes.

Theorem 3.1. *In a stable virtual age processes with asymptotically identically distributed cycles, the limiting distributions of B_t and δ_t , similar to the standard renewal processes, are given by the following equilibrium distributions:*

$$\lim_{t\to\infty} P(B_t \le x) = \lim_{t\to\infty} P(\delta_t \le x) = \frac{1}{\mu}\int_0^x R_{X_\infty}(s)ds, \tag{3.1}$$

whereas the limiting distribution of the spread Y_t is

$$\lim_{t\to\infty} P(Y_t \le x) = \frac{1}{\mu}\int_0^x s \cdot f_{X_\infty}(s)ds. \tag{3.2}$$

Proof. Denote by $m'(t)$ the generalized renewal density function for the virtual age process. Thus, $m'(t)$ can be interpreted as the rate of the corresponding point process (similar to the 'standard' renewal density, which is the rate of the ordinary renewal process). Denote by $F(x, u)$ the Cdf of a cycle that had started at the calendar time u. Then, the Cdf of B_t, denoted by $F_{B_t}(x)$, can be written as the following integral

$$F_{B_t}(x) = \begin{cases} \displaystyle\int_{t-x}^{t} \overline{F}(t-u, u)m'(u)du, & 0 \le x \le t \\ 1, \quad x > t, \end{cases} \tag{3.1.1}$$

where u is the time of the last repair before t and, accordingly, $m'(u)du$ is the probability that the cycle starts in $[u, u + du)$. The corresponding pdf is

$$f_{B_t}(x) = \begin{cases} \overline{F}(x, t-x)m'(t-x), & 0 \le x \le t \\ 0, \quad x > t. \end{cases} \tag{3.1.2}$$

Let x be fixed. Then for $t \to \infty$

$$\lim_{t\to\infty} f_{B_t}(x) = \frac{R_{X_\infty}(x)}{\mu}, \tag{3.1.3}$$

because

$$\overline{F}(x, t-x) \to_{t\to\infty} R_{X_\infty}(x), \quad \forall x \geq 0, \tag{3.1.4}$$

as the cycles converge in distribution to X_∞ and

$$m'(t-x) \to_{t\to\infty} \frac{1}{\mu}, \tag{3.1.5}$$

which results from the convergence of the cycles of the virtual age process and was shown in Dagpunar (1997) (equation 22) and Chow & Robbins (1963) (Theorem 1).

The foregoing reasoning was with respect to the backward recurrence time. The similar approach can be applied to the remaining lifetime δ_t and the spread Y_t. Thus, asymptotically, as $t \to \infty$, it is not necessary that the cycles are independent, as in the standard renewal theory, and it is sufficient that they are identically distributed. Moreover, in this case, the limiting distributions of B_t, δ_t and Y_t can be also proved similar to how it is elegantly performed in Ross (1995) using the corresponding alternating renewal process. The backward recurrence time is interpreted then as the on-time, whereas the remaining life time, as the off-time of the generalized alternating renewal process.

□

Due to the theorem 3.1 and, similar to the standard renewal theory, it holds asymptotically in our case that (the inspection paradox)

$$\forall s \in [0, \infty), \lim_{t\to\infty} R_{Y_t}(s) \geq R_{X_\infty}(s) \tag{3.3}$$

and

$$\lim_{t\to\infty} E(Y_t) = \frac{E(X_\infty^2)}{\mu}, \tag{3.4}$$

$$\lim_{t\to\infty} E(B_t) = \lim_{t\to\infty} E(\delta_t) = \frac{E(X_\infty^2)}{2\mu}. \tag{3.5}$$

Note that limiting distribution of X_∞ for the specific VA processes are given in the previous section, whereas the existence of such distributions for a general ARA_∞ model is proved, e.g., in Finkelstein (2007). It is also worth mentioning that we have conducted numerous numerical experiments (simulation) for the stable VA models considered in this paper that had also illustrated asymptotic equalities of this theorem.

4. Limiting distributions of V_t^s, V_t and V_t^e

As defined in Section 2, $F(t)$ is the Cdf of the baseline distribution and $F(t|a)$ is the Cdf of a cycle that starts with age a. The corresponding survival function is thus denoted by $\overline{F}(t|a)$, whereas the mean residual life function is $\mu_a = \int_0^\infty \overline{F}(t|a)dt$. The pdf of a cycle that starts with age a is denoted as $f(t|a) = F'(t|a)$. For stable VA processes (i.e., with asymptotically identically distributed cycles), it is important both from theoretical and practical points of view to obtain limiting distributions of the virtual age. This can be done for the two imperfect repair models considered in this paper.

4.1. ARA_∞ model

The limiting distributions of V_t^s, V_t and V_t^e for the ARA_∞ virtual age process are given by the following theorems

Theorem 4.1. *Let $V_\infty^s = \lim_{t\to\infty} V_t^s$. The asymptotic pdf of V_t^s is*

$$f_{V_\infty^s}(a) := \lim_{t\to\infty} f_{V_t^s}(a) = \frac{\mu_a}{\mu} \cdot f_{A_\infty}(a). \tag{4.1}$$

Proof. Conditioning the asymptotic distribution of B_t on V_t^s:

$$\frac{1}{\mu}\int_0^y R_{X_\infty}(s)ds = \lim_{t\to\infty} P(B_t \le y) = \int_0^\infty \lim_{t\to\infty} P(B_t \le y|V_t^s = a) f_{V_t^s}(a)da. \tag{4.1.1}$$

Fixing a, we arrive at the standard renewal process. Therefore,

$$\lim_{t\to\infty} P(B_t \le y|V_t^s = a) = \frac{1}{\mu_a}\int_0^y \overline{F}(s|a)ds. \tag{4.1.2}$$

The left hand side of equation (4.1.1) can be alternatively expressed by conditioning on the virtual age at the start of a cycle:

$$\frac{1}{\mu}\int_0^y R_{X_\infty}(s)ds = \frac{1}{\mu}\int_0^y \int_0^\infty \overline{F}(s|a) f_{A_\infty} dads, \tag{4.1.3}$$

which results in

$$\frac{1}{\mu}\int_0^y\int_0^\infty \overline{F}(s|a)f_{A_\infty}dads = \int_0^\infty\int_0^y \frac{1}{\mu_a}\overline{F}(s|a)f_{V^s_\infty}(a)dsda. \tag{4.1.4}$$

Thus, obtaining the derivatives with respect to y,

$$\frac{1}{\mu}\int_0^\infty \overline{F}(y|a)f_{A_\infty}(a)da = \int_0^\infty \frac{1}{\mu_a}\overline{F}(y|a)f_{V^s_\infty}(a)da. \tag{4.1.5}$$

Therefore, an evident solution of the pdf of V^s_∞ is

$$f_{V^s_\infty}(a) = \frac{\mu_a}{\mu}\cdot f_{A_\infty}(a). \tag{4.1.6}$$

However, the equality in integral does not guarantee the equality in the integrand. So we need to prove the uniqueness of $f_{V^s_\infty}(a)$. Assume that a function $g(t)$ is the limiting pdf of V^s_∞, with $\int_0^\infty g(t) = 1$ and

$$f_{V^s_\infty}(a) = g(a) \neq \frac{\mu_a}{\mu}\cdot f_{A_\infty}(a). \tag{4.1.7}$$

Reformulating equation (4.1.5) as:

$$\frac{1}{\mu}R_{X_\infty}(y) = \int_0^\infty \frac{1}{\mu_a}\overline{F}(y|a)g(a)da, \tag{4.1.8}$$

and considering the right hand side of above equation, we see that assuming $g(a) \neq \mu_a/\mu\cdot f_{A_\infty}(a)$ implies that the integrand could not be further simplified. Using the mean value theorem for integrals, there exists some $a^* \in [0,\infty)$ such that

$$\frac{1}{\mu}R_{X_\infty}(y) = \frac{1}{\mu_{a^*}}\overline{F}(y|a^*)\int_0^\infty g(a)da = \frac{1}{\mu_{a^*}}\overline{F}(y|a^*), \tag{4.1.9}$$

resulting in

$$\frac{\overline{F}(y|a^*)}{R_{X_\infty}(y)} = \frac{\mu_{a^*}}{\mu} = const. \tag{4.1.10}$$

This ratio cannot be a constant on all the points of $\mathbb{R}^+$ unless two survival functions are identical, which in our case, contradicts the assumption. Thus, the corresponding solution is unique, as given in (4.1). □

Interpretation of equation (4.1) depends on the baseline failure rate. For the, e.g., IFR Weibull distribution, μ_a is a decreasing function of a and satisfies:

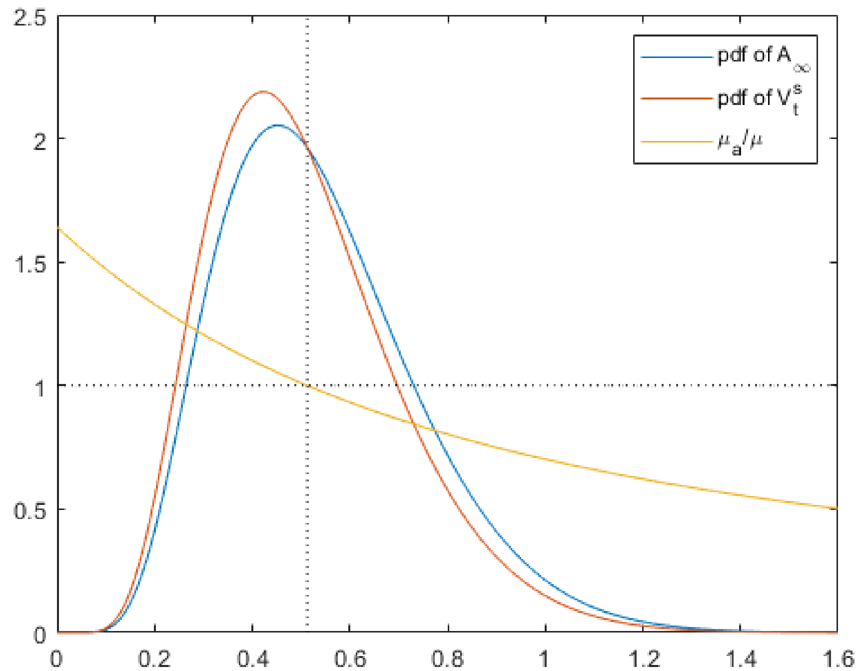

Figure 1: f_{A_∞}, $\lim_{t\to\infty} f_{V_t^s}$ and μ_a/μ, with ARA$_\infty$ configuration: $\alpha = 1, \beta = 2, \rho = 0.5$

$\mu_0 > \mu$ and $\lim_{a\to\infty} \mu_a = 0$. The pdf of V_∞^s is formed by shifting f_{A_∞} to the left, and the two curves intersect at the point $a = \tilde{A}$ with $\mu_{\tilde{A}} = \mu$. An example is showed in Figure 1.

In what follows, we are using the notion of the usual stochastic order. For convenience, its definition (Shaked & Shanthikumar (2007)) is given below.

Definition 4.1. A random variable X is said to be stochastically less (or equal to) Y, written $X \le_{ST} Y$, if the upper tail probability satisfies:

$$P(X > t) \le P(Y > t), \quad -\infty < t < \infty.$$

Remark. *Consider now some aging notions related to stable states. Finkelstein (2007) has defined the equilibrium age A^* that satisfies, if a cycle starts with age A^*, then the next cycle will also start with A^* but in expectation: $E(A_{i+1}|A_i =$*

$A^*) = A^*$. As defined previously, $\tilde{A}$ satisfies $\mu_{\tilde{A}} = \mu$, meaning that a cycle starting with age $\tilde{A}$ has the mean μ. Let $E(A_\infty)$ be the expected value of A_∞. Then, for the IFR baseline distributions,

$$\tilde{A} \leq A^* \leq E(A_\infty).$$

The proof is based on Jensen's inequality and is omitted here.

Theorem 4.2. *The limiting distributions of V_t and V_t^e are given, respectively, by*

$$\lim_{t\to\infty} P(V_t \leq y) = \frac{1}{\mu}\int_0^y \int_0^{y-a} \overline{F}(s|a) f_{A_\infty}(a) ds da, \tag{4.2}$$

$$\lim_{t\to\infty} P(V_t^e \leq y) = \frac{1}{\mu}\int_0^y \int_0^{y-a} s \cdot f(s|a) f_{A_\infty}(a) ds da. \tag{4.3}$$

Proof.

$$\begin{aligned}
\lim_{t\to\infty} P(V_t \leq y) &= \textstyle\int_0^y \lim_{t\to\infty} P(V_t \leq y | V_t^s = a) f_{V_t^s}(a) da \\
&= \textstyle\int_0^y \lim_{t\to\infty} P(B_t \leq y - a | V_t^s = a) f_{V_t^s}(a) da \\
&= \textstyle\int_0^y \frac{1}{\mu_a} \int_0^{y-a} \overline{F}(s|a) f_{V_t^s}(a) ds da \\
&= \textstyle\frac{1}{\mu} \int_0^y \int_0^{y-a} \overline{F}(s|a) f_{A_\infty}(a) ds da.
\end{aligned}$$

$$\begin{aligned}
\lim_{t\to\infty} P(V_t^e \leq y) &= \textstyle\int_0^y \lim_{t\to\infty} P(V_t^e \leq y | V_t^s = a) f_{V_t^s}(a) da \\
&= \textstyle\int_0^y \lim_{t\to\infty} P(Y_t \leq y - a | V_t^s = a) f_{V_t^s}(a) da \\
&= \textstyle\int_0^y \frac{1}{\mu_a} \int_0^{y-a} s \cdot f(s|a) f_{V_t^s}(a) ds da \\
&= \textstyle\frac{1}{\mu} \int_0^y \int_0^{y-a} s \cdot f(s|a) f_{A_\infty}(a) ds da.
\end{aligned}$$

□

4.2. Brown-Proschan model

Distinct from the ARA_∞ model, the virtual age just after the repair in the BP process is not a continuous random variable, as it has a mass in the origin (Laurent (2011)). Therefore, our results formulated in equations (4.1), (4.2) and (4.3) cannot be directly applied and the corresponding theorems should be proved in a different way. It should be also noted that it is practically important to obtain the mass for V_t^s in the origin, as it shows with what probability the last repair was perfect.

In accordance with our previous notation, let μ_0 be the mean duration of the first cycle, i.e., $\mu_0 = \int_0^\infty \overline{F}(t)dt$.

Theorem 4.3. *As t tends to infinity, V_t^s equals to 0 with probability*

$$\lim_{t\to\infty} P(V_t^s = 0) = \frac{p\mu_0}{\mu}, \tag{4.4}$$

and its density on $(0, \infty)$ is given by:

$$\lim_{t\to\infty} f_{V_t^s}(a) = \frac{\mu_a}{\mu} f_{A_\infty}(a). \tag{4.5}$$

Proof. The event $\{V_t^s = 0 \text{ as } t \to \infty\}$ is equivalent to $\{$the last repair before t as $t \to \infty$ is perfect$\}$. Let Tp be the waiting time between two perfect repairs in the corresponding BP process and F_p be its Cdf. By analogy with the alternating renewal process,

$$\lim_{t\to\infty} P(V_t^s = 0) = \frac{E(X_1)}{E(T_p)} = \frac{\mu_0}{\int_0^\infty \overline{F}_p(s)ds}. \tag{4.4.1}$$

Following Lemma 2.1 in Brown & Proschan (1983), $\overline{F}_p(t) = \overline{F}^p(t)$, leading to

$$\int_0^\infty \overline{F}_p(s)ds = \int_0^\infty e^{-p\Lambda(s)}ds = \frac{\mu}{p}, \tag{4.4.2}$$

which completes the proof of equation (4.4).

To prove equation (4.5), condition the right hand side of equation (3.1) on V_t^s:

$$\lim_{t\to\infty} P(B_t \leq y) = \lim_{t\to\infty} P(B_t \leq y | V_t^s = 0)P(V_t^s = 0) + \lim_{t\to\infty} \int_{a=0^+}^\infty P(B_t \leq y | V_t^s = a) f_{V_t^s}(a)da, \tag{4.5.1}$$

whereas conditioning of the left hand side on the virtual age at the start of a cycle, results in

$$\frac{1}{\mu}\int_0^y R_{X_\infty}(s)ds = \frac{1}{\mu}\int_0^y \overline{F}(s|0)P(A_\infty = 0)ds + \frac{1}{\mu}\int_0^y \int_{a=0^+}^\infty \overline{F}(s|a) f_{A_\infty}(a)dads. \tag{4.5.2}$$

The first term of the r.h.s of (4.5.1) is equal to that of (4.5.2), because, using the corresponding alternating renewal process,

$$\lim_{t\to\infty} P(B_t \leq y | V_t^s = 0) = \frac{E(min(y, X_1))}{E(X_1)} = \frac{\int_0^y \overline{F}(s)ds}{\mu_0}, \tag{4.5.3}$$

thus,

$$\lim_{t\to\infty} P(B_t \le y | V_t^s = 0) P(V_t^s = 0) = \frac{p}{\mu} \int_0^y \overline{F}(s) ds, \tag{4.5.4}$$

which obviously equals to the first term of the r.h.s of equation (4.5.2). This also guarantees that the second terms of the r.h.s of equations (4.5.1) and (4.5.2) are equal. Equation (4.5) can, therefore, be proved in the same way as for the ARA_∞ model.

□

Corollary 4.1. *For the Weibull baseline distribution, $\mu = E(X_\infty) = p^{1-1/\beta}\mu_0$ (Laurent (2011)), which results in*

$$\lim_{t\to\infty} P(V_t^s = 0) = p^{\frac{1}{\beta}}. \tag{4.6}$$

Theorem 4.4. *The limiting distributions of V_t and V_t^e are given by:*

$$\lim_{t\to\infty} P(V_t \le y) = \frac{p}{\mu} \int_0^y \overline{F}(s) ds + \frac{1}{\mu} \int_{0^+}^y \int_0^{y-a} \overline{F}(s|a) f_{A_\infty}(a) ds da, \tag{4.7}$$

$$\lim_{t\to\infty} P(V_t^e \le y) = \frac{p}{\mu} \int_0^y s \cdot f(s) ds + \frac{1}{\mu} \int_{0^+}^y \int_0^{y-a} s \cdot f(s|a) f_{A_\infty}(a) ds da, \tag{4.8}$$

respectively.

Proof. Consider first V_t.

$$\lim_{t\to\infty} P(V_t \le y) = \lim_{t\to\infty} P(V_t \le y | V_t^s = 0) P(V_t^s = 0) + \lim_{t\to\infty} \int_{0^+}^y P(V_t \le y | V_t^s = a) f_{V_t^s}(a) da. \tag{4.7.1}$$

The first term of the right hand side of the above equation can be further developed as:

$$\begin{aligned}
&\lim_{t\to\infty} P(V_t \le y | V_t^s = 0) P(V_t^s = 0) \\
&= \lim_{t\to\infty} P(B_t \le y | V_t^s = 0) \lim_{t\to\infty} P(V_t^s = 0) \\
&= \tfrac{1}{\mu_0} \textstyle\int_0^y \overline{F}(s) ds \cdot \tfrac{\mu_0}{\mu} p \\
&= \tfrac{p}{\mu} \textstyle\int_0^y \overline{F}(s) ds,
\end{aligned}$$

whereas the second term of the right hand side of (4.7) can be derived in the same way as for the ARA_∞ model.

Consider now V_t^e.

$$\lim_{t\to\infty} P(V_t^e \le y) = \lim_{t\to\infty} P(V_t^e \le y | V_t^s = 0) P(V_t^s = 0) + \int_{0^+}^{y} \lim_{t\to\infty} P(V_t^e \le y | V_t^s = a) f_{V_t^s}(a) da. \tag{4.8.1}$$

The first term of the right hand side of the above equation can be further developed as:

$$\begin{aligned}
&\lim_{t\to\infty} P(V_t^e \le y | V_t^s = 0) P(V_t^s = 0) \\
&= \lim_{t\to\infty} P(Y_t \le y | V_t^s = 0) \lim_{t\to\infty} P(V_t^s = 0) \\
&= \tfrac{1}{\mu_0} \int_0^y s \cdot f(s) ds \cdot \tfrac{p\mu_0}{\mu} \\
&= \tfrac{p}{\mu} \int_0^y s \cdot f(s) ds,
\end{aligned}$$

whereas the second term of the right hand side of (4.8) can be derived in the same way as for the ARA_∞ model.

□

5. Application in maintenance: Optimal degree of imperfect repair

In this section, we focus particularly on the repair process of the ARA_∞- type with the increasing baseline failure rate. Finkelstein (2015) has considered the optimal degree of imperfect repair that achieves the minimal, expected long-run cost rate for the repaired accordingly system. In the following, we shall make two extensions of this optimization problem. But first, let us recall the setting.

Assume that the cost of an imperfect maintenance action at any cycle depends only on the degree of repair ρ. Denote by $C(\rho)$ this cost. It is natural to assume that it is an increasing function of ρ and

$$C_m = C(0) \le C(\rho) \le C(1) = C_p, \tag{5.1}$$

where C_m and C_p are the costs of minimal and perfect repairs, respectively. Consider now the long-run average maintenance cost rate. The ARA_∞ process enters its steady state and the mean cycle length for the corresponding Weibull IFR baseline distribution, $\mu(\rho) = E(X_\infty | \alpha, \beta, \rho)$, is an increasing function of ρ

and can be obtained by integrating (2.7). Based on the renewal reward theory reasoning, the expected long-run cost per unit of time c_ρ is given by:

$$c(\rho) = \frac{C(\rho)}{\mu(\rho)}. \tag{5.2}$$

Assume a rather flexible functional form for $C(\rho)$

$$C(\rho) = C_m + (C_p - C_m)\rho^u, \quad u > 0. \tag{5.3}$$

Existence of an optimal maintenance degree ρ^*, which minimizes the long-run average cost rate $c(\rho)$ has been addressed in Finkelstein (2015). Basically, it requires that $c(\rho)$ be increasing as ρ tends to 1.

5.1. Recycling: reward based on backward recurrence time or on virtual age at retirement

The above reasoning considers the expected cost for an infinite horizon. In practice, however, systems are not operating forever, i.e., a 'retirement' threshold T_r is often predefined in a way that once the total working time exceeds T_r, system's operation is terminated (and it is usually replaced by a new one). The replaced system can be sometimes recycled and the corresponding gain is generated in accordance with its condition, i.e., the better the condition, the larger the gain. A typical example is the garage of the used cars, where the status of the used car, as well as its accidents history, are carefully examined to determine an appropriate price.

The optimization problem is formulated as follows: a system is under imperfect repair and is planned to 'retire' at time T_r. The expected lifetime in the steady-state regime, $\mu(\rho)$ is a function of the repair degree ρ and satisfies $\mu(1) << T_r$. The cost of the repair actions is defined by equation (5.3). At time T_r, the system is recycled and a reward, R_w, is assigned based on the state variable Y that can either be the backward recurrence time, B_{T_r} or the virtual age, V_{T_r}. Assume that the relation between the reward and Y is given by the following functional form:

$$R_w = re^{-\nu Y}, \quad r > 0, \nu > 0, \tag{5.4}$$

where r defines the maximal reward that could be obtained if the system is in the state "Good as New" at T_r. Under the condition $\mu(1) << T_r$, the expected long-run cost per unit of time can, therefore, be defined, for instance, as

$$c_r(\rho) = \frac{C(\rho)}{\mu(\rho)} - \frac{r}{T_r} E(e^{-\nu Y}), \tag{5.5}$$

We are now interested in obtaining the optimal degree of repair ρ_r^* that minimizes the cost rate defined by equation (5.5). In practice, the corresponding decisions can be made either based on the virtual age of a system or on the time since the last repair (backward recurrence time). The latter, although giving less information on the state of a system, can be easier obtained, whereas V_t needs more information on the history of the repair process, which often can be unavailable. For ordinary renewal processes, these quantities are the same, whereas for the imperfect repair process they are, obviously, different.

5.1.1. Reward based on backward recurrence time: $Y = B_{T_r}$

Since $T_r >> \mu(1)$, the distribution of the backward recurrence time at retirement can be described by equation (3.1). The expected long-run cost rate is, therefore,

$$c_r(\rho | Y = B_{T_r}) = \frac{C(\rho)}{\mu(\rho)} - \frac{r}{T_r \mu(\rho)} \int_0^\infty e^{-\nu x} R_{X_\infty}(x|\rho) dx. \tag{5.6}$$

Consider the effect of a repair efficiency ρ on the expected value of the reward R_w. It can be seen from the definition (5.4) that for the IFR baseline distributions, this expected value is decreasing when ρ is increasing (as the cycles of the corresponding steady-state virtual age process are stochastically increasing with ρ). Thus, given the parameters of the model, there can exist an optimal degree of repair that minimizes (5.6). This is illustrated by the lowest curve in Fig.2

5.1.2. Reward based on virtual age: $Y = V_{T_r}$

Assume now the reward is defined according to the virtual age V_{T_r} at the retirement time. As $T_r >> \mu(1)$, the virtual age tends to its asymptotic value and its distribution is given by equation (4.2). Given the Weibull baseline distribution, the expected long-run cost rate is then defined as

$$c_r(\rho|Y = V_{T_r}) = \frac{C(\rho)}{\mu(\rho)} - \frac{r}{T_r\mu(\rho)} \int_0^\infty \int_0^x e^{-\nu x - \alpha x^\beta + \alpha a^\beta} f_{A_\infty}(a|\rho) da dx. \quad (5.7)$$

It can be shown that the virtual age V_t as t tends to infinity is decreasing in ρ. Specifically, it tends to infinity when ρ tends to 0 (minimal repair) and then decreases to the asymptotic virtual age of the ordinary renewal process (which is just the corresponding backwards recurrence time) when ρ tends to 1 (perfect repair). This behavior *dramatically differs* from that for B_{T_r} in the previous subsection (it was increasing in ρ), which is a meaningful fact. Eventually, it results in a larger optimal value of ρ than that defined by the cost rate function (5.6). Moreover, the optimal ρ for the case without recycling (see (5.2)) lies between these two values (Fig.2). The following numerical example illustrates our reasoning.

Example 5.1. *Let* $\alpha = 1$, $\beta = 3$. *Thus, the baseline survival function is* $R(t) = e^{-t^3}$. *Let* $C_p = 1$, $C_m = 0.3$, $u = 4$. *Then the long-run expected cost per unit of time without the recycling reward, is plotted by the solid line in Fig2. The optimal repair degree* $\rho^* \approx 0.57$ *with a minimal expected cost* $c(\rho^*) = 0.6966$.

Consider now the reward policy defined as $T_r = 20$, $r = 20$, $\nu = 2$. *The expected maximal reward defined by* r/T_r *equals one, and is of the same order of magnitude as* C_p. *It is, therefore, necessary to take into account the reward when optimizing the maintenance degree. When the reward is based on the backward recurrence time at retirement,* $\rho_r^* = 0.52$ *with the corresponding expected cost rate* $c(\rho_r^*) = 0.1363$ *(dashed line) and when the reward is given according to the corresponding virtual age, then* $\rho_r^* = 0.65$ *with the expected cost rate* $c(\rho_r^*) = 0.4195$ *(dash-dotted line). These results are consistent with our previous analysis showing that* B_t *and* V_t *have an opposite impact on the value of* ρ_r^*.

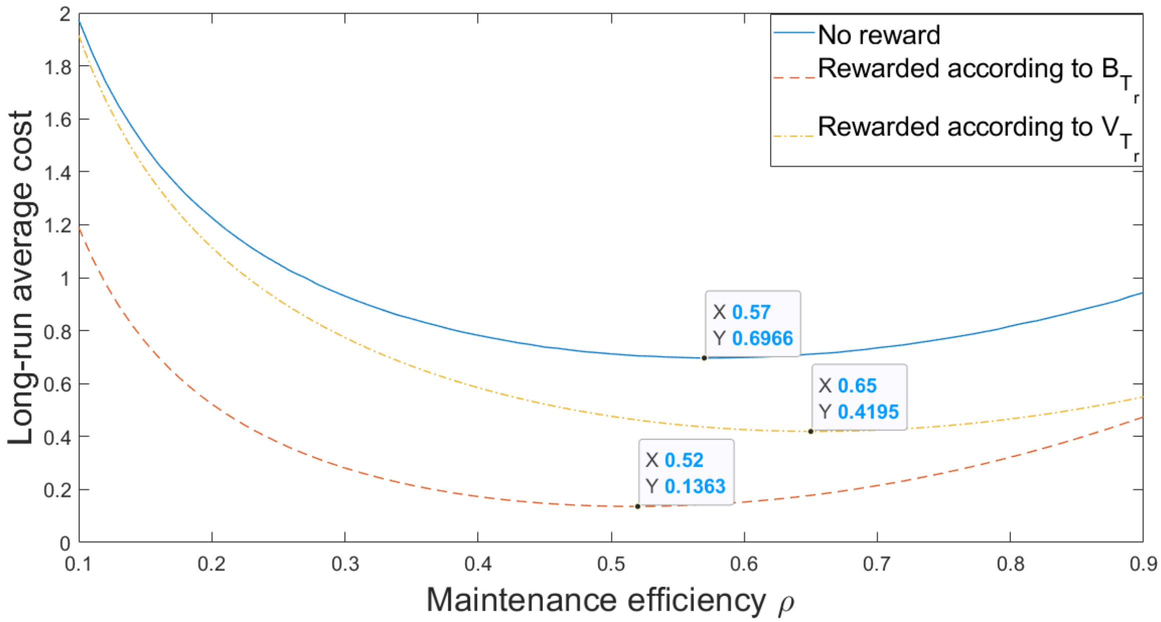

Figure 2: Optimal repair degrees

5.2. Minimization of the long-run expected failure frequency of a series system under constrained budget

In this section, as another example, we deal with the constrained optimal imperfect maintenance problem (Pham & Wang (1996)) of the following type. We consider a series system of n independently operating and instantaneously maintained/repaired components with imperfect repair of the ARA$_\infty$ type. The first interval of the repair process of the ith component is Weibull-distributed with parameters (α_i, β_i), accordingly. The repair degrees of each component, $\vec{\rho} = \{\rho_1, \rho_2 ... \rho_n\}$ form the vector of decision variables. The repair cost, $C_i(\rho_i)$ depends only on the repair degree and is independent from the initial lifetime distribution. It is defined by equation (5.3) with different parameters for components. The expected long-run repair cost per unit of time of the system must not exceed the predefined cost threshold, C_{max}, i.e.,

$$\sum_{i=1}^{n} \frac{C_i(\rho_i)}{\mu(\alpha_i, \beta_i, \rho_i)} \leq C_{max}, \tag{5.8}$$

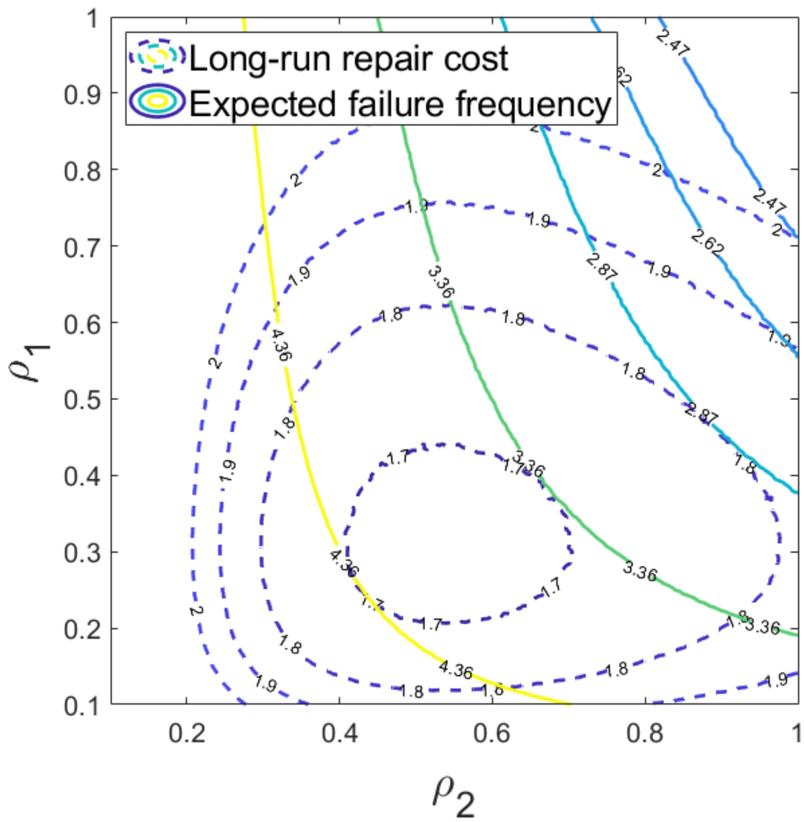

Figure 3: Optimal repair degrees determined by the contours that are tangent to each other

By the "long-run", as previously, we mean the steady-state case, therefore, the denominator $\mu(\alpha_i, \beta_i, \rho_i)$ is the mean duration of the asymptotic cycle of the component i given α_i, β_i and ρ_i.

Under the constraint (5.8), we would like to minimize the steady-state failure frequency for the system. Thus, the corresponding objective function is defined as:

$$\lambda_s = \sum_{i=1}^{n} \frac{1}{\mu(\alpha_i, \beta_i, \rho_i)}. \tag{5.9}$$

In the following, for illustration, we will consider the simplest case of two components in series that differ only in the shape parameter β. Assume also that the cost function is the same for each component, i.e. $C_1(\rho) = C_2(\rho) = C(\rho)$.

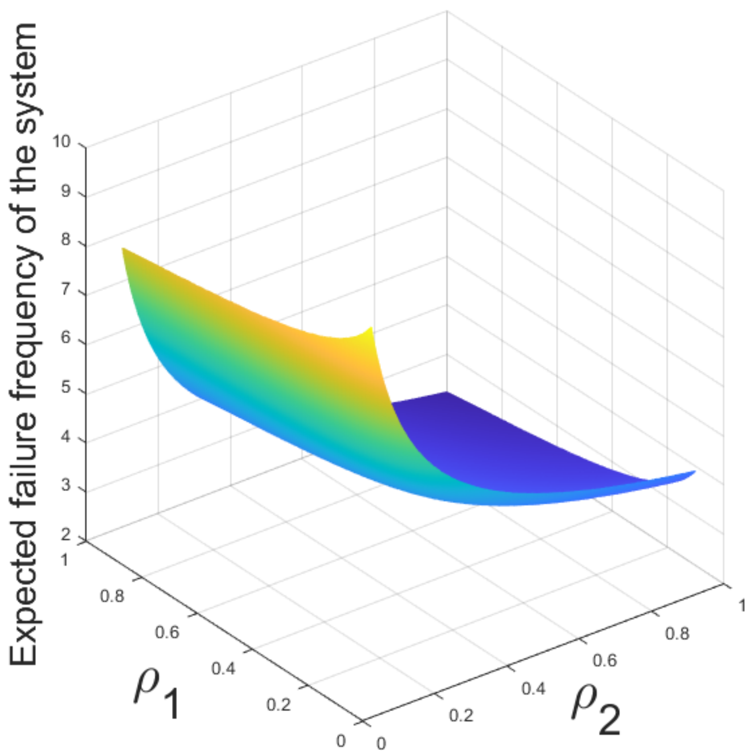

Figure 4: Expected failure frequency of the system

Example 5.2. *Parameters of the Weibull distributions for the components are chosen as* $\alpha_1 = \alpha_2 = 1$, $\beta_1 = 1.5$, $\beta_2 = 3$, *whereas parameters of the cost function are:* $C_p = 1$, $C_m = 0.3$ *and* $u = 2$. $\vec{\rho} = (\rho_1, \rho_2)$ *is the decision vector. Obviously, without the constraint, the optimal repair degree is just* $\vec{\rho^*} = (1, 1)$. *However,when the maintenance cost threshold* C_{max} *is not large enough, we may not have enough resources to perform perfect repairs.*

The optimal repair degrees vector $\vec{\rho^*} = (\rho_1^*, \rho_2^*)$ *is defined by the points where the contour of the expected cost and that of the system's expected failure frequency are tangent to each other (figure 3). When, for instance,* $C_{max} =$

$[1.7, 1.8, 1.9, 2.0]$, *the corresponding optimal degrees are*

$$(\rho_1^*, \rho_2^*) = [(0.39, 0.67), (0.47, 0.89), (0.58, 0.98), (0.71, 1)],$$

accordingly, and the resulting minimal expected failure frequency of the series system are $[3.36, 2.87, 2.62, 2.47]$. *Additionally, Fig 4 shows the corresponding pattern for the expected failure frequency of the system in the unconstrained case (no costs involved).*

6. Concluding remarks

This paper studies asymptotic distributions for stable virtual age processes. We first show that the limiting distributions of the backward recurrence time, the remaining lifetime and the spread that characterize an ordinary renewal process can be generalized to the case of the virtual age processes with asymptotically identically distributed cycles. Then we derive new analytical expressions for all limiting distributions of interest. We also discuss the importance of the age reduction mechanism for the obtained results. The provided examples highlight the practical value of our findings in reliability engineering.

We plan to continue with this topic in the future in several directions. For instance, asymptotic distributions in stable virtual age models involving imperfect preventive maintenances can be considered. A typical example is the ARA_1CM-ARA_∞PM process described in Doyen et al. (2019): corrective maintenances of the ARA_1 type are unable to keep the repaired system in a steady state, whereas stationarity can be achieved by the periodic PMs of the ARA_∞ type. Therefore, it could be of interest to look at the asymptotic distribution of the virtual age just after the PM in this case. Limiting distributions in other imperfect maintenance models such as the Arithmetic reduction of intensity with infinite memory (ARI_∞) model (Doyen & Gaudoin (2004)) can also be worth of further investigation.

Another related typical problem of interest is the statistical inference using the incomplete failure history. The repair process is observed at an arbitrary time t when a system has already entered its steady state. Our results give the distribution of the virtual age at the start of observation, and therefore, can result in better parameter estimation.

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