Complexity analysis of energy-efficient single machine scheduling problems

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Abstract

This paper deals with the complexity analysis of several energy-oriented single-machine scheduling problems addressed in the literature. The considered machine may be in different states: OFF, ON, Idle, or in transitions between them. The energy consumption of the machine at each time-slot is state-dependent. The objective is the minimization of the total energy consumption costs over the planning horizon.

For this purpose, two particular cases with constant energy price and increasing energy prices during all the time-slots are studied. These two problems are proved to be polynomial. Moreover, the general version of this problem with Time-Of-Use (TOU) energy prices and different processing times of the jobs is investigated in two versions: with and without the fixed sequence for the jobs. As the results, the version with the fixed sequence is proved to be polynomial, and the version without the fixed sequence (general version) is proved to be NP-hard.

This paper also introduces different lower bounds to deal this general version of the problem. The performances of these lower bounds are discussed based on different numerical instances.

Keywords: Finite states single-machine scheduling; Energy efficiency; Time of use electricity tariffs; Complexity analysis

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1. Introduction

Nowadays, a significant part of the energy in each country is consumed in the industry. For example, about 30% of all the end-use energy consumption in the United States is associated with industrial activities ([1]). It is well-known that most countries use electricity as the main energy source for manufacturing. Rising electricity prices in addition to the ecological considerations have encouraged researchers to study the efficiency improvement of a production system in terms of energy consumption and costs involved, to reduce energetic production costs and environmental impact.

The energy consumption of a manufacturing system can be minimized at different levels such as machine-level, product-level, and system-level. Contrary to the machine-level and product-level, which need great financial investments to redesign the machine(s) or product(s), at the system-level, manufacturers may reduce their energy consumptions using the existing decision tools based on optimization techniques. In this paper, some energy-efficient scheduling problems are studied to optimize the energy consumption of a single machine manufacturing system.

Based on the literature review analysis, the total energy consumptions and total energy costs minimization are the two objectives mostly used for dealing with the energy efficient scheduling problems. In the case of a single machine, the total energy consumptions consist of the amount of energy consumed during non-processing states (NPE) (e.g. the start-up, the transition between different states, shut down states, and idle states), and during processing state (PE). Therefore, decision makers may focus on the NPE or PE parts of any system to reduce its energy consumption. For this purpose, one of the most usual approaches is investigating the NPE consumption and using a scheduling method to change the job's processing order and the machine's state within a production shift.

In the following, a summary of the few papers addressing energy efficient scheduling problems on the single machine systems is given.

[2] presented a literature review of decision support models for energy efficient production planning. For each machine, the amount of its energy consumption depends on one or several factors, e.g. type of the machine, the machine's state, processing speed, and type of the jobs. Among the papers which consider the state factor, [3] developed operational methods by using some dispatching rules. They also proposed a multi-objective model to minimize the energy consumption and total completion time of the system. [4] presented a framework for a system with idle and setup states to minimize total energy consumption and total tardiness simultaneously. Energy consumption and total completion time minimization of a single machine are studied in [5], using a multi-objective genetic algorithm and dominance rules. [6] developed a model and algorithm that minimize energy consumption in a single machine production system with decision whether the machine should be idle or switched on or off between consecutive jobs. [7] considered a single-machine scheduling problem with power-down mechanism. The aim is to find an optimal processing sequence of jobs and determine if the machine execute a power-down operation between two consecutive jobs that minimize both total energy consumption and maximum tardiness. [8] addressed a single-machine scheduling problem with cumulative deteriorating effect and multiple maintenance activities to determine the sequence of jobs and the number of maintenance activities as well as their positions, in order to minimise energy consumption. To solve this problem, a mixed integer linear programming model in addition to a genetic algorithm (GA), a particle swarm optimisation (PSO) algorithm and a hybrid PSO (HPSO) approach are proposed. [9] studied a bi-objective single machine scheduling problem with energy consumption constraints, in which the objective functions were the total weighted completion time and the total weighted tardiness. They adopted a multi-objective particle swarm optimization algorithm to solve this problem.

Among the papers which consider the speed factor, [10] examined the trade-off between total energy consumption and total weighted tardiness in a single machine environment with sequence-dependent setup times, where different jobs can be operated at varying speed levels. [11] and [12] studied the complexity of the deadline-based preemptive and non-preemptive scheduling problems with a variable processing speed. The scheduling problems with continuous resource and energy constraint are addressed in [13], [14] and [15], to minimize the amount of energy consumption.

In addition to the mentioned factors which change the amount of energy consumption of a machine, different policies are also considered by researchers to investigate the possible modifications on the total energy costs of a system, such as time-of-use pricing (TOU), real-time pricing (RTP), and critical peak pricing (CPP). For example, [16] addressed an energy-conscious scheduling problem of a single machine, in which each processing job has its power consumption, and electricity prices may vary from hour to hour throughout a day. [17] proposed a method for energy efficient and labor-aware production scheduling at the unit process level under real-time electricity pricing. [18] studied a single machine scheduling problem which deals with the assignment of a set of jobs to available time periods under time-varying electricity pricing, while considering requested due dates of jobs so as to minimize total penalty costs for earliness and tardiness of jobs and total energy consumption costs, simultaneously. [19] worked on a non-preemptive single-machine scheduling problem under TOU electricity tariffs in order to minimize the total tardiness and total energy cost. They proposed a mixed-integer multi-objective mathematical programming model and several new holistic genetic algorithms for this problem.

[20] developed a new greedy insertion heuristic algorithm with a multi-stage filtering mechanism for single machine scheduling problems under TOU electricity tariffs. [21] proposed an energy-efficient algorithm to minimize the total flow time and the total

cost of a single machine scheduling problem, when the processor has variable speeds and different energy consumptions. Generic mixed-integer programming models for a single machine scheduling that minimize total energy cost at volatile energy prices are presented in [22] and [23]. Some scheduling problems with arbitrary power demands for the jobs, and uniform or variable processing speeds in preemptive and non-preemptive cases are studied in [1] to minimize total electricity cost under a time of use electricity tariffs. A preemptive scheduling problem with energy constraint in each time-slot, different energy consumption for each job, and the electricity time-varying prices are investigated in [24] to minimize the total electricity consumption costs and the operations postponement penalty costs.

[25] deals with a single machine scheduling problem which has different possible states. They proposed a mathematical model to minimize total energy consumption costs with variable energy prices. The same problem as [25], is considered in [26] to improve the previous mathematical model. They also presented a new mathematical model to obtain the optimal schedule for the machine's state and the job's sequence, simultaneously. Then, a new heuristic algorithm and a genetic algorithm are proposed in [27] to solve the problem without the fixed sequence for the jobs. The complexity of a preemptive multi-states single machine scheduling problem is analyzed in [28], using a dynamic programming approach. [29] addressed a new production scheduling method to minimize the total energy costs, when a finite set of states (multiple idle modes) is considered for the machine.

A comprehensive literature analysis demonstrates that there are several energy-efficient single machine scheduling problems, but to the best of our knowledge, there are a few studies which deal with the complexity of this kind of problem, when the machine has finite states. This paper aims to fill this gap in the literature. The remainder of the paper is organized as follows. In section 2, the problem statement and its assumptions are introduced. In section 3, the complexity of the fixed sequence case of the problem under Time-Of-Use energy costs is investigated. Then, in sections 4, the complexity of two variants of the scheduling problems are analyzed. Section 5 studies the complexity of the general problem with Time-Of-Use energy costs, and presents some lower bounds for this problem. Finally, section 6 summarizes the contributions of this study and draws some future directions for next studies.

2. Problem statement

The addressed problem can be described as follows. Let consider n jobs which must be scheduled on a single machine within a given planning horizon (T time-slots). The jobs must be processed non-preemptively. All the jobs are available at time-slot 0 to T ($r_j = 0 \quad ; \forall j = 1, ..., n$).

The machine has 3 main states (ON, OFF, Idle) and 2 transition states for turning on and turning off (Ton, Toff). When the machine is in state $k \in \{OFF,\ Ton,$ $ON,\ Toff,\ Idle\}$, it must remain in the same state during a fixed number of time-slots (d_k). For example, a transition from ON to OFF implies that, the machine must stay in Toff state during $d_{Toff} = \beta_2$ time-slots. In other words, switching ON and OFF the processor causes delays. Each state k is also characterized by an energy consumption (e_k). This means the energy consumption of the machine in state ON is constant and independent from the processed job (Fig. 1). The machine must be in OFF state during the initial ($t = 0$) and the final ($t = T$) time-slots. Note that, in this study, time-slot 0 is just for identifying the initial state of the machine which is OFF. The scheduling horizon is from time-slot 1 to T. Without loss of generality, the following relations are also considered for the machine states energy consumption:

$$e_{ON} > e_{Idle} > e_{OFF} = 0 \tag{1}$$

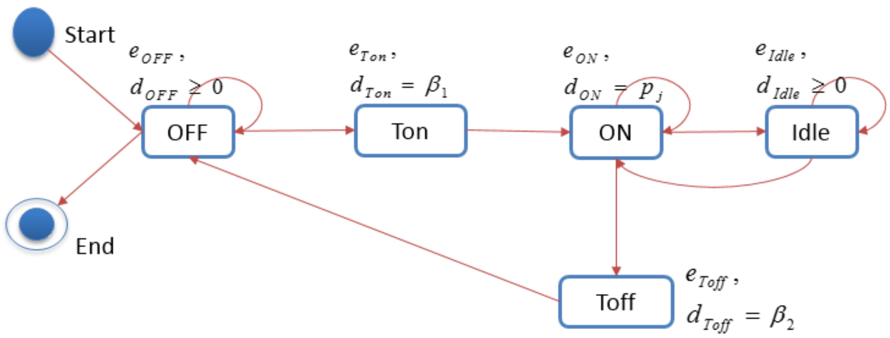

Figure 1: Machine states and possible transitions.

$$e_{Ton} > e_{OFF} = 0 \tag{2}$$

$$e_{Toff} > e_{OFF} = 0 \tag{3}$$

The minimum energy consumption of the machine is during state OFF, which is considered negligible ($e_{OFF} = 0$). These assumptions and the possible transitions between different states are illustrated in Fig. 1. In this paper, c_t ;$\forall t = 1, ..., T$, indicates the unit of energy price at time-slot t. Moreover, φ_k represents the set of time-slots' number in which the machine is in state $k \in \{OFF,\ Ton,\ ON,\ Toff,\ Idle\}$, and F represents the objective value of any feasible solution. Besides, F^* and φ_k^* are related to the optimal solution. The objective value for each solution of this problem may be computed with the following formulation:

$$\begin{aligned} F = (e_{OFF} \times \textstyle\sum_{t\in\varphi_{OFF}} c_t) + (e_{Ton} \times \sum_{t\in\varphi_{Ton}} c_t) + \\ (e_{ON} \times \textstyle\sum_{t\in\varphi_{ON}} c_t) + (e_{Toff} \times \sum_{t\in\varphi_{Toff}} c_t) + (e_{Idle} \times \sum_{t\in\varphi_{Idle}} c_t) \end{aligned} \tag{4}$$

Since the initial and final states of the machine are assumed as OFF states, the machine is in Ton/Toff state at least for once. Let consider $\lambda \in N$ the number of turning on or turning off over the T time-slots, then:

$$|\varphi_{Ton}| = \lambda \times \beta_1 \tag{5}$$

$$|\varphi_{Toff}| = \lambda \times \beta_2 \tag{6}$$

Moreover, for each additional Toff/Ton transitions, the machine must stay in OFF state during at least one time-slot ($|\varphi_{OFF}| \geq \lambda$).

The required number of time-slots to have a feasible solution for this problem is equal to sum of the processing times ($P = \sum_{j=1}^{n} p_j$), plus the required number of time-slots for initial Ton and final Toff states ($\beta_1 + \beta_2$), plus one (because the machine must be in OFF state at the end of the horizon). In this problem, one of the essential conditions to have at least one feasible solution is that the T value must be always larger than the number of required time-slots. The difference between the T value and the required time-slots' value, can be defined as the number of extra time-slots. Let x indicates the number of these extra time-slots, then:

$$x = T - P - (\beta_1 + \beta_2 + 1) \tag{7}$$

For example, in a problem with 3 jobs, 15 time-slots and the parameters' values as: $\beta_1 = 2,\ \beta_2 = 1,\ p_1 = 2,\ p_2 = 1,\ p_3 = 2$, x is equal to $[15 - 5 - (2+1+1)] = 6$. Based on the problem's objective, the machine must be put into the non-processing states during these x time-slots (one or several cases among initial or final OFF states, idle states between the ON states, and middle-OFF states may be used). Note that each middle-OFF state consists of a sequence of Toff, OFF during at least one time-slot, and Ton states.

Therefore, the cardinal of sets $\varphi_k\ \forall k \in \{Ton,\ ON,\ Toff,\ Idle,\ OFF\}$ in any feasible

t	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	Cost
c_t	0	8	8	8	4	4	4	3	3	3	2	2	2	2	2	10	10	10	10	3	3	3	2	2	2	2	6	6	3	3	3	5	5	
ON							16	12	12	12	8	8	8	8	8							12	8	8	8	8								
OFF	0	0	0	0													0	0	0									0	0	0	0	0	0	
Idle																																		
Turn on					20	20														15	15													
Turn off																10											6							
The schedule	Off				Ton		Job1			Job2		Job3				Toff	Off			Ton		Job4		Job5			Toff	Off						222

Figure 2: An example of the general problem

solution are as follow:

$$\begin{cases} |\varphi_{Ton}| = \lambda \times \beta_1 \\ |\varphi_{ON}| = P \\ |\varphi_{Toff}| = \lambda \times \beta_2 \\ |\varphi_{Idle}| \geq 0 \\ |\varphi_{OFF}| \geq \lambda \end{cases} \tag{8}$$

where:

$$|\varphi_{Ton}| + |\varphi_{ON}| + |\varphi_{Toff}| + |\varphi_{Idle}| + |\varphi_{OFF}| = T \tag{9}$$

Since for each feasible solution, we have $|\varphi_{ON}| = P$, so:

$$|\varphi_{Ton}| + |\varphi_{Toff}| + |\varphi_{Idle}| + |\varphi_{OFF}| = T - P \tag{10}$$

For example, the gantt chart for an instance of 5 jobs and 32 time-slots with the parameters' values as: $\beta_1 = 2,\ \beta_2 = 1,\ p_1 = 3,\ p_2 = 2,\ p_3 = 4,\ p_4 = 2,\ p_5 = 3$ and $e_{OFF} = 0,\ e_{ON} = 4,\ e_{Idle} = 2,\ e_{Ton} = 5,\ e_{Toff} = 1$ is provided in Fig. 2.

As can be seen, for this solution $\lambda = 2$ ($|\varphi_{OFF}| = 12,\ |\varphi_{Ton}| = 4,\ |\varphi_{ON}| = 14,\ |\varphi_{Toff}| = 2,\ |\varphi_{Idle}| = 0$).

In the next sections, first of all the complexity analysis of a specific version of the problem with a pre-determined order for the jobs which is investigated in [25] is addressed. By using Graham's three fields notation, this problem can be defined as $1,TOU|sequence,\ states|TEC$, where TEC represents our objective which is Total Energy consumption Costs and TOU represents Time-Of-Use energy costs. Then, the complexities of several problems without a pre-determined order are also analyzed when there exists a regular trend for the energy prices during two consecutive time-slots, and when the energy prices are irregular. For this purpose, three different problems such as: $(1, c_t = c|states|TEC)$, $(1, c_t < c_{t+1}|states|TEC)$, and $(1, TOU|states|TEC)$ are studied.

3. The problem with time-of-use (TOU) energy prices and fixed sequence

Problem $1,TOU|sequence,\ states|TEC$ is already addressed by [25] in the literature, where they proposed an LP mathematical model to find the optimal schedule for the system with a fixed sequence of the jobs by making decisions at the machine level. Their experimental results proved the disability of the proposed analytical solution to solve the instances of this problem with more than 60 jobs during 3 hours. Moreover, their research was based on the fact that "since the shop floor scheduling problem is considered to be an NP-hard-complete problem, so, this problem cannot be solved in real life.", and they proposed a genetic algorithm to find a solution for any instance of this problem.

In this paper, before considering the more general problems, we want to give a prove for the complexity of this problem which is not addressed in the literature. For this purpose, in the following a dynamic programming approach is presented to model this problem ($1,TOU|sequence,\ states|TEC$). This approach is based on a finite graph whose dimension (number of vertices and edges) is dependent on the total number of processing times (P) and the total number of time-slots (T).

In what follows, this approach it has been described in more details.

3.1. Graph construction steps

The using graph consists of several decision-making levels (l) and nodes, where each level represents one time-slot of the problem's horizon. As a consequence, the graph consists of $T+1$ decision levels ($0 \leq l \leq T$), and each level has some nodes. Let us consider H_l to present the possible nodes for level l, which corresponds to the possible states of the machine in each time-slot. Each node of is also characterized by the cumulative number of production units (k) from time-slot 0 to l. Because of the problem's assumption that machine is in OFF state within the initial and final time-slots, $H_0 = \{I\}$ and $H_T = \{F\}$. In this graph, I represents that the machine is in initial state and it did not performed any job, and F indicates the final status of the machine after processing all the jobs (P).

Let us explain our approach by using an example with 3 jobs, 15 time-slots, 5 production units and the parameters' values as Table 1, which is presented in Figure 3. For this instances, the possible number of production units at time-slot (level) 7 can be 1 to 5 units ($H_7 = \{1,2,3,4,5\}$). Because, if it were less than 1 unit, there will not be enough time to complete all the jobs during the rest of horizon. Moreover, according to the set up times, 7 units of time are not enough to process all the 5 production units and turn off the machine. For these reasons, a time interval is defined for each node which represents the earliest and the latest possible levels that contains node k in the graph, and is shown by $\tau_k = \{l_{min(k)}, \cdots, l_{max(k)}\}$. The time interval time for node k simplifies as:

$$\tau_k \in \{\beta_1 + k, \cdots, x + \beta_1 + k\} \quad ; \forall k \in \{1, \cdots, P\} \tag{11}$$

Therefore, the first step among the graph construction steps is to place all the nodes ($k \in \{I, 1, \cdots, P, F\}$) in the graph using their related time interval (τ_k).

Once all the possible nodes of the graph are placed, the second step is to draw the edges and compute their value which represent the total energy cost for doing the related transitions between node (k,l) and node (k',l') ($Ev_{(k,l)-(k',l')}$; $\forall\ k \in H_l,\ k' \in H_{l'},\ l' \geq l+1$).

In order to distinguish the different types of the edges, they are divided into three main sets (E_1, E_2, E_3). The first set (E_1) connects the nodes with the same k number between level l and $l+1$, which indicates the Idle state with the edge value of:

$$Ev_{(k,l)-(k,l+1)} = c_{l+1} \times e_{Idle} \quad ;\forall k \in \{p_1, p_1+p_2, \cdots, \sum_{j=1}^{n-1} p_j\} \tag{12}$$

where c_l is the unite of energy price in time-slot l, e_{Idle} is the machine's energy consumption in $Idle$ state, and p_j is the process time of job j. The total number of edges for this set is: $|E_1| = (n-1) \times x$.

The second set (E_2) connects nodes (k,l) to node $(k+1,l')$ that consists of three cases. The first case is for the initial turning on phase of the system, with the edge value as:

$$Ev_{(I,0)-(1,l)} = \sum_{i=l-\beta_1}^{l-1} (c_i \times e_{Ton}) + c_l \times e_{ON} \tag{13}$$

the second case is for processing the next production unit with the edge value as:

$$Ev_{(k,l)-(k+1,l')} = c_{l'} \times e_{ON} \quad ;l' = l+1 \tag{14}$$

and the third case is for the final turning off phase of the system, with the edge value as:

$$Ev_{(P,l)-(F,T)} = \sum_{i=l+1}^{l+\beta_2} (c_i \times e_{Toff}) + \sum_{i=l+\beta_2+1}^{T} (c_i \times e_{OFF}) \tag{15}$$

The cardinal of this set of edges is equal to:

$$|E_2| = (P+1) \times (x+1) \tag{16}$$

Table 1: Parameters' values of instance (3,15)

State	Power consumption (kW)	Required period
ON	6	5={2,1,2}
OFF	0	-
Idle	2	-
Turn on	8	2
Turn off	1	1

The third set (the last one E_3), shows the middle shutdown between two processing states. It connects node (k,l') with node $(k+1,l)$, where, $l' \in \{l_{min(k)},\cdots,x+k-\beta_2-1\}$, and the edge value is:

$$
\begin{aligned}
&Ev_{(k,l')-(k+1,l)} = \\
&\textstyle\sum_{i=l'+1}^{l'+\beta_2}(c_i \times e_{Toff}) + \sum_{i=l-\beta_1}^{l-1}(c_i \times e_{Ton}) + c_l \times e_{ON} \\
&\quad ;\forall k \in \{p_1, p_1+p_2, \cdots, \textstyle\sum_{j=1}^{n-1} p_j\}
\end{aligned}
\tag{17}
$$

The cardinal of this set is:

$$|E_3| = \sum_{i=1}^{x-(\beta_1+\beta_2)} i \times (n-1) \tag{18}$$

As a result, the related graph of a problem with T time-slots, P production units and x value has the total number of nodes and edges as follow:

$$|V| = P \times (x+1) + 2 \cong TP \tag{19}$$

$$|E| = |E_1| + |E_2| + |E_3| \cong T^2 P \tag{20}$$

For example, the corresponding graph of our instance with $P=5, T=15, \beta_1=2, \beta_2=1, x=6$, and different energy prices (c_t), consists of 37 nodes and 66 edges (see Fig. 3).

period	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
cost	0	3	2	5	4	2	3	4	7	2	5	4	6	1	3	2

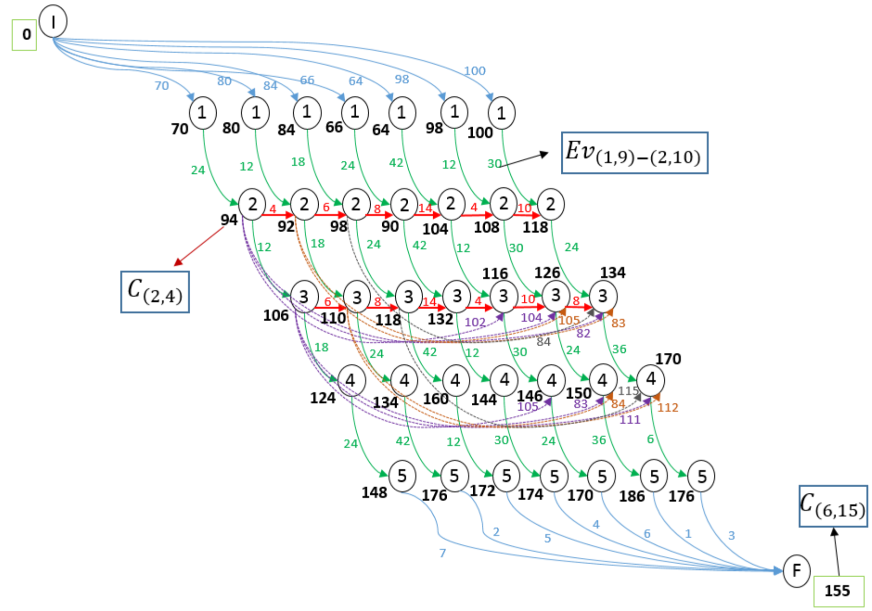

Figure 3: The related graph for instance (3,15)

3.2. Complexity analysis

Regarding to the modeling approach for this graph, each path which passed from node $(I,0)$ to node (F,T) presents a feasible solution for the problem, and the shortest one (with the minimum sum of the edges' values) represents the optimal solution. Let us consider cost $C_{(k,l)}$ associated to node $k \in H_l$ which indicates the minimum cost for performing k production units within l time-slots and it has a positive value. The recurrence relationship to obtain each node's cost is as follows, where $A_{k,l}$ is the set of the precedent nodes that are connected to node (k,l) directly:

$$
\begin{aligned}
&C_{(I,0)} = 0 \\
&C_{(k,l)} = \min_{(k',l') \in A_{k,l}} \{C_{(k',l')} + Ev_{(k',l')-(k,l)}\}
\end{aligned}
\tag{21}
$$

For the presented instances in Figure 3 we have $A_{4,11} = \{(3,10),(3,6),(3,5)\}$.

As a consequence, by this approach, $C_{(F,T)}$ represents the objective value of the optimal solution for the problem.

Since with the presented approach any instance of the considered problem can be modeled by using a finite graph, if the shortest path of this graph (the optimal solution) can be also obtained in a polynomial time, then we can conclude that the problem is polynomial. On this account, Dijkstra's algorithm, which is one of the most efficient algorithms to find the shortest path between the source node and every other node of a graph, is used in this study. The worst case implementation of this algorithm runs in $O(|E|+|V|\log|V|)$ ($|E|$: number of the edges and $|V|$: number of the nodes) which is based on a min-priority queue ([30]). As a consequence, the complexity of this algorithm for the presented problem is equal to:

$$O(T^2P + TP\log TP) \cong O(T^2P) \tag{22}$$

Since the largest possible value of P is T (worst case analysis), it means that Dijkstra's algorithm obtain the optimal solution of this problem with T time-slots in $O(T^3)$ which is a polynomial time.

As a result, it can be concluded that unlike what the authors considered in [25], problem $1,TOU|sequence,\ states|TEC$ is polynomial.

The application of Dijkstra's algorithm on the considered instance, is presented in Figure 3. To be more clear, the best solution for this instance is to turn the machine on from time-slot 0 and process all the jobs based on their order during time-slots 3 to 7, and finally, turning the machine off in time-slot 9 which has the cost of 155.

After that we succeeded to prove that problem $1,TOU|sequence,\ states|TEC$ is polynomial, for the next step, we are interested to analyse the complexities of several problems when the jobs' sequence is not fixed ($1|states|TEC$). For this purpose, the problems with and without a regular trend for the energy prices are studied in the next sections.

4. The problems with a regular trend of the energy prices

In this section, the complexities of the problems with a constant, increasing, and decreasing energy prices for the case without the fixed sequence are investigated.

4.1. $pb_2 : 1, c_t = c|states|TEC$

Theorem 1. *If the energy price during the horizon time is constant ($c_t = c; \forall t = 1, ..., T$), the problem ($1, c_t = c|states|TEC$) is polynomial.*

Proof. In this problem (pb_2), the price of energy during all the time-slots is constant ($c_t = c\ ; \forall t = 1, \cdots, T$), so, for any feasible solution of pb_2, the expression of the objective function, denoted by F_2, may be deduced from Equation (4) as:

$$\begin{aligned} F_2 = [(e_{OFF} \times |\varphi_{OFF}|) + (e_{Ton} \times |\varphi_{Ton}|) + \\ (e_{ON} \times |\varphi_{ON}|) + (e_{Toff} \times |\varphi_{Toff}|) + (e_{Idle} \times |\varphi_{Idle}|)] \times c \end{aligned} \tag{23}$$

Let consider the solution $S_2^\star$ such that: $|\varphi_{OFF}^{2\star}| = T - (\beta_1 + P + \beta_2), |\varphi_{Ton}^{2\star}| = \beta_1, |\varphi_{ON}^{2\star}| = P, |\varphi_{Toff}^{2\star}| = \beta_2, |\varphi_{Idle}^{2\star}| = 0$, with the objective function value of $F_2^\star$. For any other feasible solution of pb_2, as S_2^i with objective function F_2^i, the relation between F_2^i and $F_2^\star$ is as follow:

$$\begin{aligned} F_2^i - F_2^\star = [(|\varphi_{OFF}^i| - |\varphi_{OFF}^{2\star}|) \times e_{OFF} + (|\varphi_{Ton}^i| - |\varphi_{Ton}^{2\star}|) \times e_{Ton} + \\ (|\varphi_{Toff}^i| - |\varphi_{Toff}^{2\star}|) \times e_{Toff} + (|\varphi_{Idle}^i| - |\varphi_{Idle}^{2\star}|) \times e_{Idle}] \times c \end{aligned} \tag{24}$$

Regarding equations (1), (2), and (3), we have $e_{Ton} = e_{OFF} + \delta_1$, $e_{Toff} = e_{OFF} +$

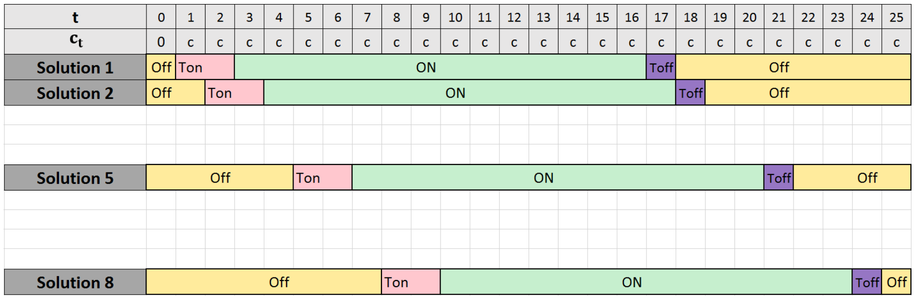

Figure 4: The possible optimal solutions of pb_2 for an example with $T = 25, P = 14, \beta_1 = 2, \beta_2 = 1$

$\delta_2,\ e_{Idle} = e_{OFF} + \delta_3,\ e_{ON} = e_{OFF} + \delta_4$ with $(\delta_1, \delta_2, \delta_3, \delta_4 > 0)$, thus:

$$\begin{aligned} F_2^i - F_2^\star &= [(|\varphi^i_{OFF}| + |\varphi^i_{Ton}| + |\varphi^i_{Toff}| + |\varphi^i_{Idle}| - T + P) \times e_{OFF} \\ &+ (|\varphi^i_{Ton}| - \beta_1) \times \delta_1 + (|\varphi^i_{Toff}| - \beta_2) \times \delta_2 + (|\varphi^i_{Idle}| - 0) \times \delta_3] \times c \\ &= [(|\varphi^i_{Ton}| - \beta_1) \times \delta_1 + (|\varphi^i_{Toff}| - \beta_2) \times \delta_2 + |\varphi^i_{Idle}| \times \delta_3] \times c \end{aligned} \tag{25}$$

Based on equation (8) and the fact that $\lambda \geq 1$, we have $|\varphi^i_{Ton}| - \beta_1 \geq 0,\ |\varphi^i_{Toff}| - \beta_2 \geq 0,\ |\varphi^i_{Idle}| \geq 0$. Consequently, it can be concluded that $F_2^i - F_2^\star \geq 0$ which means $F_2^\star$ is a lower bound of this problem (pb_2). Since $S_2^\star$ is also a feasible solution for pb_2, for this reason, $S_2^\star$ is the optimal solution. Note that, in this problem, $S_2^\star$ is not a unique optimal solution. All feasible solutions which have the same value as $|\varphi_k^{2^\star}|$ for state $k \in \{OFF, Ton, ON, Toff, Idle\}$, have the same objective function value. Moreover, there is not any priority between the jobs of this problem.

As a consequence, the optimal solution of problem pb_2 is when the machine has just one turning on and one turning off states, and processes all the jobs (in any order) continuously without any idle state. Also, it remains in OFF state during the rest of the horizon. For example, for the presented problem in Fig. 4, there exist 8 different solutions with the same objective value. Any of these solutions can be considered as the optimal solution. Since this set of optimal solutions can be obtained directly, pb_2 is polynomial.

□

4.2. $pb_3 : 1, c_t < c_{t+1}|states|TEC$

Theorem 2. *If the energy prices are increasing between two consecutive time-slots ($c_t < c_{t+1}; \forall t = 1, \cdots, T-1$) and $e_{OFF} = 0$, the problem ($1, c_t < c_{t+1}|states|TEC$) is polynomial.*

Proof. Total energy consumption costs minimization of a production system can be reached by two ways: energy consumptions minimization and/or total energy costs minimization. In this problem (pb_3), unlike the previous one, the energy costs are different in each time-slot, for this reason, both of these ways may be used. The total energy consumptions of the machine should be minimized by minimizing the number of time-slots for each state (as it is demonstrated for pb_2). Moreover, the total energy costs may be minimized by placing the high-consumption states at the low-cost time-slots.

Let consider the solution $S_3^\star$ as:

$$\begin{cases} \varphi_{Ton}^{3\star} = \{1, \cdots, \beta_1\} \\ \varphi_{ON}^{3\star} = \{\beta_1 + 1, \cdots, \beta_1 + P\} \\ \varphi_{Toff}^{3\star} = \{\beta_1 + P + 1, \cdots, \beta_1 + P + \beta_2\} \\ \varphi_{OFF}^{3\star} = \{\beta_1 + P + \beta_2 + 1, \cdots, T\} \\ |\varphi_{Idle}^{3\star}| = 0 \end{cases} \tag{26}$$

with the objective function value of $F_3^\star$ which may be computed from equation (4). For any other feasible solution S_3^i of pb_3 with F_3^i as the objective value, the relation between F_3^i and $F_3^\star$ is as follow:

$$F_3^i - F_3^\star = e_{OFF} \times (\sum_{t\in\varphi_{OFF}^i} c_t - \sum_{t\in\varphi_{OFF}^{3\star}} c_t) + e_{Ton} \times (\sum_{t\in\varphi_{Ton}^i} c_t - \sum_{t\in\varphi_{Ton}^{3\star}} c_t) + \\ e_{ON} \times (\sum_{t\in\varphi_{ON}^i} c_t - \sum_{t\in\varphi_{ON}^{3\star}} c_t) + e_{Idle} \times \sum_{t\in\varphi_{Idle}^i} c_t + e_{Toff} \times (\sum_{t\in\varphi_{Toff}^i} c_t - \sum_{t\in\varphi_{Toff}^{3\star}} c_t) \tag{27}$$

The other possible solutions for this problem can be divided into two main sets. The first set case is composed of the solutions obtained by adding some non-processing states (Idle or middle-off) between two processing states, and the second one is obtained by changing the starting time of processing, adding some initial-off states. All the other solutions are mixed of these two cases.

For the first case, regarding equations (1), (2), and (3), obviously adding some non-processing states which consume more than OFF state ($e_{OFF} = 0$), causes an increase of the total energy consumptions and consequently the total energy consumption costs. Let consider a general example (Fig. 5) such that: $1 < t_1 < t_2 < t_3 < t_4 < T$. If between two ON states, the machine goes to the Idle state during time-slot $t_2 + 1$, based on the equation (27), we have:

$$F_3^i - F_3^\star = e_{Idle} \times c_{t_2+1} + e_{ON} \times (\sum_{t=t_2+2}^{t_3+1} c_t - \sum_{t=t_2+1}^{t_3} c_t) + \\ + e_{Toff} \times (\sum_{t=t_3+2}^{t_4+1} c_t - \sum_{t=t_3+1}^{t_4} c_t) + e_{OFF} \times (\sum_{t=t_4+2}^{T} c_t - \sum_{t=t_4+1}^{T} c_t) \tag{28}$$

Therefore,

$$F_3^i - F_3^\star = e_{Idle} \times c_{t_2+1} + e_{ON} \times (c_{t_3+1} - c_{t_2+1}) + \\ + e_{Toff} \times (c_{t_4+1} - c_{t_3+1}) - e_{OFF} \times c_{t_4+2} \tag{29}$$

Since in pb_3, $e_{OFF} = 0$, $e_{Idle} > 0$, and $\forall t' > t;\ c_{t'} > c_t$, we have:

$$F_3^i - F_3^\star \geq 0 \tag{30}$$

By the same procedure, it can be proved that multiple shut down operations (middle-off states) which includes a sequence of Toff, OFF and Ton states with at least $[(e_{Toff} \times$

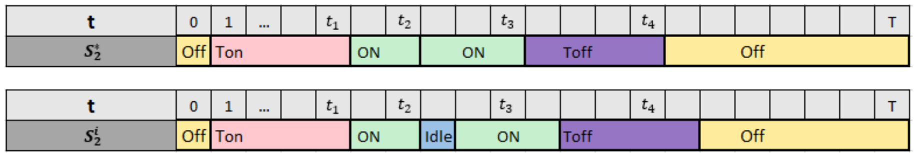

Figure 5: The comparison between solution S_3^i and $S_3^\star$ of problem pb_3: case1

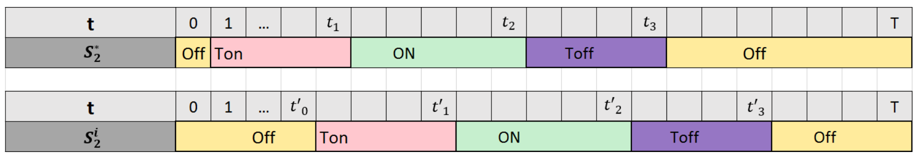

Figure 6: The comparison between solution S_3^i and $S_3^\star$ of problem pb_3: case2

$\beta_2) + (e_{OFF} \times 1) + (e_{Ton} \times \beta_1)]$ energy consumption units, causes increase of the total energy consumptions.

For the second case, let consider a general example (Fig. 6) such that: $1 < t_1 < t_2 < t_3 < T$ and $t_z < t'_z \quad \forall z = 1, 2, 3$. Based on the equation (27), we have:

$$F_3^i - F_3^\star = e_{OFF} \times (\textstyle\sum_{t=1}^{t'_0} c_t + \sum_{t=t'_3+1}^{T} c_t - \sum_{t=t_3}^{T} c_t) + e_{Ton} \times (\sum_{t=t'_0+1}^{t'_1} c_t - \sum_{t=1}^{t_1} c_t) + \\ e_{ON} \times (\textstyle\sum_{t=t'_1+1}^{t'_2} c_t - \sum_{t=t_1+1}^{t_2} c_t) + e_{Toff} \times (\sum_{t=t'_2+1}^{t'_3} c_t - \sum_{t=t_2+1}^{t_3} c_t) \tag{31}$$

Accordingly,

$$F_3^i - F_3^\star = e_{OFF} \times (\textstyle\sum_{t=1}^{t'_0} c_t - \sum_{t=t_3+1}^{t'_3} c_t) + e_{Ton} \times (\sum_{t=t'_0+1}^{t'_1} c_t - \sum_{t=1}^{t_1} c_t) + \\ e_{ON} \times (\textstyle\sum_{t=t'_1+1}^{t'_2} c_t - \sum_{t=t_1+1}^{t_2} c_t) + e_{Toff} \times (\sum_{t=t'_2+1}^{t'_3} c_t - \sum_{t=t_2+1}^{t_3} c_t) \tag{32}$$

Regarding to equations (1), (2), and (3), since in pb_3, $e_{OFF} = 0$ and $\forall t' > t;\ c_{t'} > c_t$, we have:

$$F_3^i - F_3^\star \geq 0 \tag{33}$$

Thus, for any feasible solution as S_3^i, we have $F_3^i - F_3^\star \geq 0$. It means that $F_3^\star$, is a feasible lower bound of this problem (pb_3), and $S_3^\star$ is the optimal solution.

So, in the optimal solution, the machine must be in Ton state from time-slot 1 to β_1.

t	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
c_t	0	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26
Solution 1	Off	Ton		ON														Toff	Off							

Figure 7: The optimal solution for an instance of problem with $c_t < c_{t+1} \quad (\forall t = 1, \cdots, T-1)$

t	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
c_t	0	27	26	25	24	23	22	21	20	19	18	17	16	15	14	13	12	11	10	9	8	7	6	5	4	3
Solution 8	Off								Ton		ON														Toff	Off

Figure 8: The optimal solutions for an instance of problem with $c_t > c_{t+1} \quad (\forall t = 1, \cdots, T-1)$

Then, the jobs must be processed from time-slot $\beta_1 + 1$ to $\beta_1 + 1 + P$ in any order, and finally, the machine must be in turning off and OFF states consecutively (Fig. 7). Therefore, pb_3 is a polynomial problem with the optimal objective value of $F_3^\star$. □

Note that, with the same approach, but in a backward way, it can be proved that: If the energy prices between two consecutive time-slots were decreasing ($c_t > c_{t+1}; \forall t = 1, \cdots, T-1$), the problem is also polynomial (Fig. 8).

5. The problem with time-of-use (TOU) energy prices without fixed sequence

For the third part of this paper, the problem with TOU energy prices without fixed sequence is addressed (Fig. 2). In the following the complexity of this problem, when the jobs have different processing times, is investigated.

5.1. $pb_4 : 1, TOU|states|TEC$

Theorem 3. *If the jobs have different processing times, the Problem* $(1, TOU|states|$ *TEC) is strongly NP-hard.*

Proof. The proof is based on the fact that the decision problem related to this optimization problem, may be reduced to a 3-PARTITION problem, which is strongly NP-hard ([31]). In the following, the same approach which is utilised by [1] to prove that the problem with just two states for the machine (ON-OFF) and arbitrary power demands for the jobs is NP-hard, is used for our problem.

Given positive integers $\{a_1, a_2, \cdots, a_{3t}, b\}$, such that:

$$b/4 < a_j < b/2 \quad ;\forall j = 1,2,\cdots,3t \tag{34}$$

$$\sum_{j=1}^{3t} a_j = tb \tag{35}$$

The following instance of pb_4 (equations (36) to (39)), with $n = 3t$ jobs and $T = tb + t + 3$ time-slots can be constructed. The machine consumes the units of energy just when it is in state ON ($e_{ON} \neq 0$). Moreover, the unit of energy price in some time-slots (tb time-slots) equals to 0, and for the rest ($t+3$ time-slots) is equal to c ($c > 0$):

$$p_j = a_j \quad ;\forall j = 1,2,\cdots,3t \tag{36}$$

$$c_t = c \quad ;\forall t = 0,1,2,tb+t+3,(i+1)b+i+3 \quad ;\forall i = 0,1,\cdots,t-1 \tag{37}$$

$$c_t = 0 \quad ;\forall t = i(b+1)+3,\cdots,i(b+1)+(b-1)+3 \quad ;\forall i = 0,1,\cdots,t-1 \tag{38}$$

$$e_{OFF} = e_{Ton} = e_{Idle} = e_{Toff} = 0, \quad e_{ON} = E \tag{39}$$

Let us consider a decision problem that searches a solution with the total energy consumption costs equal to 0. A schedule with total energy costs of 0 ($TEC = 0$), exists if and only if, the machine is in one of the states that consume 0 unit of energy during the time-slots with $c_t = c$, and it is in state ON when $c_t = 0$. This can be achieved if and only if, all the $3t$ jobs are partitioned over the t intervals with the length of b time-slots.

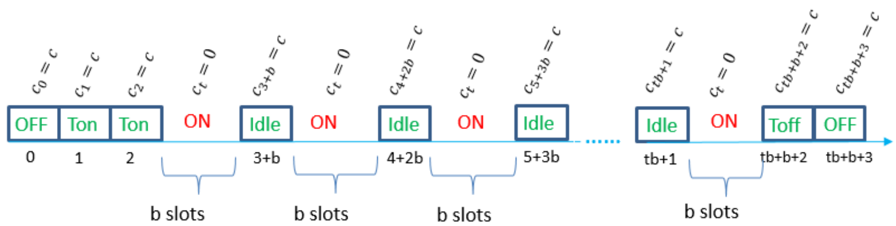

Figure 9: An example of pb_4 which transfers to a 3-PARTITION problem

For this purpose, the $3t$ jobs must be partitioned to t sets such that each set consists of 3 jobs, and the sum of their processing times must be equal to b. Then, each set must be partition into one interval with the length of b time-slots, which can be achieved if and only if, 3-PARTITION has a solution (see Fig. 9). Therefore, since the 3-PARTITION is known as an NP-complete problem ([31]), as a consequence, pb_4 is NP-hard. $\square$

Since it is not possible to find the optimal solution of an NP-hard problem by using the usual exact methods, approximation methods are developed to find a near optimal feasible solution for this kind of problems ([27]). A usual tool to evaluate the performances of such methods is to propose lower bounds. For this reason, in the following we attempted to propose some lower bounds for problem pb_4.

5.2. Lower bounds for pb_4

From the given set of the time-slots' energy cost $C = \{c_t\ ; \forall t = 1, \cdots, T\}$, let consider the set $\tilde{C} = \{\tilde{c}_1, \tilde{c}_2, \cdots, \tilde{c}_T\}$, which contains the time-slots' energy cost in the increasing order, such that $\tilde{c}_1 \le \tilde{c}_2 \le \cdots \le \tilde{c}_T$. Then, the following relation can also be written:

$$\sum_{t=1}^{\theta} \tilde{c}_t \le \sum_{t=1}^{\theta} c_t \quad \forall \theta = 1, \cdots, T \tag{40}$$

Regarding equations (1), (2), and (3), the OFF state has the minimum energy consumption between all the non-processing states. Therefore, in the cases that the unite energy prices are increasing, obviously adding some non-processing states which consume more than OFF state would increase the total energy consumptions and consequently the total energy consumption costs. That is why during the rest of this study, for defining the lower bounds, the minimum number of required time-slots are considered for Ton and Toff states, and states of the machine during all the remaining time-slots are considered as OFF state.

Let define LB_1 as the cost obtained allocating the cheapest time-slots to each state. So, we have:

$$LB_1 = (e_{OFF} \times \sum_{t=1}^{T-(\beta_1+P+\beta_2)} \tilde{c}_t) + (e_{Ton} \times \sum_{t=1}^{\beta_1} \tilde{c}_t) + (e_{ON} \times \sum_{t=1}^{P} \tilde{c}_t) + (e_{Toff} \times \sum_{t=1}^{\beta_2} \tilde{c}_t) \quad (41)$$

Lemma 4. *LB_1 is a lower bound of pb_4.*

Proof. Regarding the equation (4), the optimal value of total energy costs ($F_4^\star$) for the problem pb_4 can be computed as:

$$\begin{aligned} F_4^\star = e_{OFF} \times \textstyle\sum_{t \in \varphi_{OFF}^{4\star}} c_t + e_{Ton} \times \sum_{t \in \varphi_{Ton}^{4\star}} c_t + \\ e_{ON} \times \textstyle\sum_{t \in \varphi_{ON}^{4\star}} c_t + e_{Toff} \times \sum_{t \in \varphi_{Toff}^{4\star}} c_t + e_{Idle} \times \sum_{t \in \varphi_{Idle}^{4\star}} c_t \end{aligned} \quad (42)$$

Based on the problem's assumption (equations (8),(9),(10)), the cardinal of $\varphi_k^\star$ for each state in the optimal solution are as follows ($\lambda^\star \geq 1$):

$$\begin{cases} |\varphi_{Ton}^{4\star}| = \lambda^\star \times \beta_1; \quad |\varphi_{ON}^{4\star}| = P; \\ \\ |\varphi_{Toff}^{4\star}| = \lambda^\star \times \beta_2; \quad |\varphi_{Idle}^{4\star}| \geq 0; \quad |\varphi_{OFF}^{4\star}| \geq \lambda^\star \end{cases} \quad (43)$$

$$|\varphi_{Ton}^{4\star}| + |\varphi_{Toff}^{4\star}| + |\varphi_{Idle}^{4\star}| + |\varphi_{OFF}^{4\star}| = T - |\varphi_{ON}^{4\star}| = T - P \quad (44)$$

Based on equation 40, we have the following equations:

$$\begin{cases} \sum_{t=1}^{\beta_1} \tilde{c}_t \leq \sum_{t=1}^{|\varphi_{Ton}^{4\star}|} \tilde{c}_t \leq \sum_{t\in\varphi_{Ton}^{4\star}} c_t \\ \\ \sum_{t=1}^{P} \tilde{c}_t \leq \sum_{t\in\varphi_{ON}^{4\star}} c_t \\ \\ \sum_{t=1}^{\beta_2} \tilde{c}_t \leq \sum_{t=1}^{|\varphi_{Toff}^{4\star}|} \tilde{c}_t \leq \sum_{t\in\varphi_{Toff}^{4\star}} c_t \\ \\ \sum_{t=1}^{T-(\beta_1+P+\beta_2)} \tilde{c}_t \leq \sum_{t=1}^{|\varphi_{OFF}^{4\star}|} \tilde{c}_t \leq \sum_{t\in\varphi_{OFF}^{4\star}} c_t \end{cases} \tag{45}$$

Based on the above equations (45), the following relation can be obtained for LB_1 and $F_3^\star$:

$$LB_1 - F_4^\star \leq 0 \tag{46}$$

Therefore, LB_1 is a lower bound for pb_4. □

To define the first lower bound (LB_1), the non-preemption and precedence constraints for the states of the machine (Fig. 1), and the fact that the machine must be in one and only one state per time-slot are relaxed. Only the importance of energy price in each time-slot is considered. For example, by these constraints, if the machine starts to process job j in time-slot t, the machine must be in ON state from time-slot t to $t + p_j - 1$, and it is not possible to be in other states during them. For this purpose, we defines the second lower bound (LB_2), which sorts the time-slots based on their energy costs and allocates them into Ton, ON, $Toff$, and OFF states, respectively and continuously. By this way, in the second lower bound's solution the machine has only one state in each time-slot, but, the non-preemption and the precedence constraints for the states are relaxed yet. LB_2 is computed as follow:

$$\begin{aligned} LB_2 = (e_{Ton} \times \textstyle\sum_{t=1}^{\beta_1} \tilde{c}_t) + (e_{ON} \times \sum_{t=\beta_1+1}^{\beta_1+P} \tilde{c}_t) + \\ (e_{Toff} \times \textstyle\sum_{t=\beta_1+P+1}^{\beta_1+P+\beta_2} \tilde{c}_t) + (e_{OFF} \times \sum_{t=\beta_1+P+\beta_2+1}^{T} \tilde{c}_t) \end{aligned} \tag{47}$$

Lemma 5. *LB_2 is a lower bound of pb_4.*

Proof. pb_4 with $\tilde{c}_t$;$\forall t = 1, \cdots, T$, converts to pb_3 which its optimal solution is provided in section 4.2. Accordingly, the optimal solution of pb_3 ($F_3^{\star}$) may be used as a lower bound of pb_4. As it is proved during equations (27) to (33), $LB_2 \leq F_3^i \leq F_4^{\star}$. For this reason, LB_2 is a lower bound of problem pb_4. □

Let consider $\underline{C}_j$ that computes the minimum cost of performing job j ($\forall j = 1, ..., n$) non-preemptively during its possible time-slots. As it is explained before, the possible time-slots that the machine can be in Ton, Toff and ON states depend to the total number of time-slots in the horizon, the number of extra time-slots, and the number of required time-slots for performing each job. Thus, $\underline{C}_j$ may be formulated as follow.

$$\begin{aligned} &\underline{C}_j = \min\{c_t + c_{t+1} + \cdots + c_{t+p_j-1}\}; \\ &\forall t \in \{\beta_1 + 1, \cdots, T - p_j - \beta_2\}; \quad \forall j \in \{1, \cdots, n\} \end{aligned} \tag{48}$$

Then, following the same idea, the minimum costs for Ton, Toff and OFF states are obtained with the following formulations.

$$\underline{C}_{Ton} = \min\{c_t + c_{t+1} + \cdots + c_{t+\beta_1-1}\}; \quad \forall t \in \{1, 2, \cdots, x+1\} \tag{49}$$

$$\underline{C}_{Toff} = \min\{c_t + c_{t+1} + \cdots + c_{t+\beta_2-1}\}; \quad \forall t \in \{T - x - \beta_2, \cdots, T - \beta_2\} \tag{50}$$

$$\underline{C}_{OFF} = \sum_{t=1}^{T-(\beta_1+P+\beta_2)} \tilde{c}_t \tag{51}$$

As it has been discussed before, regarding equations (1), (2), and (3), the Idle state consumes more than OFF state, that is why, all the remaining non-processing states are considered as OFF states. The idea of the third lower bound named LB_3 is to allocate each state to its minimum costs possible $\underline{C}_j$ $(\forall j \in \{1,\cdots,n\})$, $\underline{C}_{Ton}$, $\underline{C}_{Toff}$, $\underline{C}_{OFF}$:

$$LB_3 = (e_{OFF} \times \underline{C}_{OFF}) + (e_{Ton} \times \underline{C}_{Ton}) + (e_{ON} \times \sum_{j=1}^{n} \underline{C}_j) + (e_{Toff} \times \underline{C}_{Toff}) \quad (52)$$

Lemma 6. *LB_3 is a lower bound of pb_4.*

Proof. To evaluate $\underline{C}_j; \forall j \in \{1,\cdots,n\}$, the constraint that the machine can process one job per time-slot is relaxed and the processing order for the jobs is not considered. Moreover, to evaluate $\underline{C}_j$ $(\forall j \in \{1,\cdots,n\})$, $\underline{C}_{Ton}$, $\underline{C}_{Toff}$, $\underline{C}_{OFF}$, the constraints that the machine must be in just one state per time-slot, and the relationship between different state of the machine are relaxed. On this account, for a feasible solution of pb_4 we have the following relations:

$$\underline{C}_{OFF} \leq \sum_{t \in \varphi^{4\star}_{OFF}} c_t; \quad \underline{C}_{Ton} \leq \sum_{t \in \varphi^{4\star}_{Ton}} c_t; \quad \underline{C}_{Toff} \leq \sum_{t \in \varphi^{4\star}_{Toff}} c_t; \quad \sum_{j=1}^{n} \underline{C}_j \leq \sum_{t \in \varphi^{4\star}_{ON}} c_t \quad (53)$$

And we have:

$$\begin{cases} e_k \times \underline{C}_k & \leq e_k \times \sum_{t \in \varphi^{4\star}_k} c_t \quad ; \forall k \in \{OFF,\ Ton,\ Toff\} \\ \\ e_{ON} \times \sum_{j=1}^{n} \underline{C}_j & \leq e_{ON} \times \sum_{t \in \varphi^{4\star}_{ON}} c_t \end{cases} \quad (54)$$

Consequently:

$$LB_3 \quad \leq F_3^{\star} \quad (55)$$

Therefore, LB_3 is a lower bound of this problem. □

Moreover, the optimal solution of the preemption version of this problem can be defined as LB_4.

Lemma 7. *LB_4 is a lower bound of pb_4.*

Proof. As it is demonstrated in a previous work ([28]), the preemption version of this

Table 2: Energy consumption profile of a machine. ([25])

States and transitions	Power consumption	required time-slots
ON	4 kW	$\sum$ process times
OFF	0 kW	-
Idle	2 kW	-
Toff	1 kW	1
Ton	5 kW	2

problem $(1, TOU|states, pmtn|TEC)$ which is a subproblem of pb_4, is polynomial. As a consequence, it's optimal solution may be used as the fourth lower bound (LB_4) of this problem.

□

5.3. Numerical experiments for the proposed lower bounds

To evaluate the efficiency of the proposed lower bounds in this study, several randomly generated instances are considered. Based on the presented examples in a pervious study ([25]), the machine setup data for all the examined instances in this study are identical and considered as Table. 2.

For each size of the problem, ten instances have been examined. To generate the instances, the unit of energy price in each time-slot, as well as the processing times of the jobs are randomly generated between $[1,10]$ and $[1,5]$, respectively. Table. 3 represents the gap between the objective value of each lower bound and the obtained optimal solution by CPLEX software in percentage. These results are presented for the problem smaller than $(35,209)$ size problem, because the CPLEX software was not able to find the optimal solution for the instances larger than this size during 3 hours or 10800 seconds time limitation. The numerical results have been illustrated by minimum, average and maximum obtained gap value for each problem size. The results show that between LB_1, LB_2, and LB_3, in all the cases LB_2 proposed a better average gap. As can be seen, among these lower bounds, LB_4 which is the obtained optimal solution of the preemptive case of this problem by CPLEX, presents the solutions that are more near to

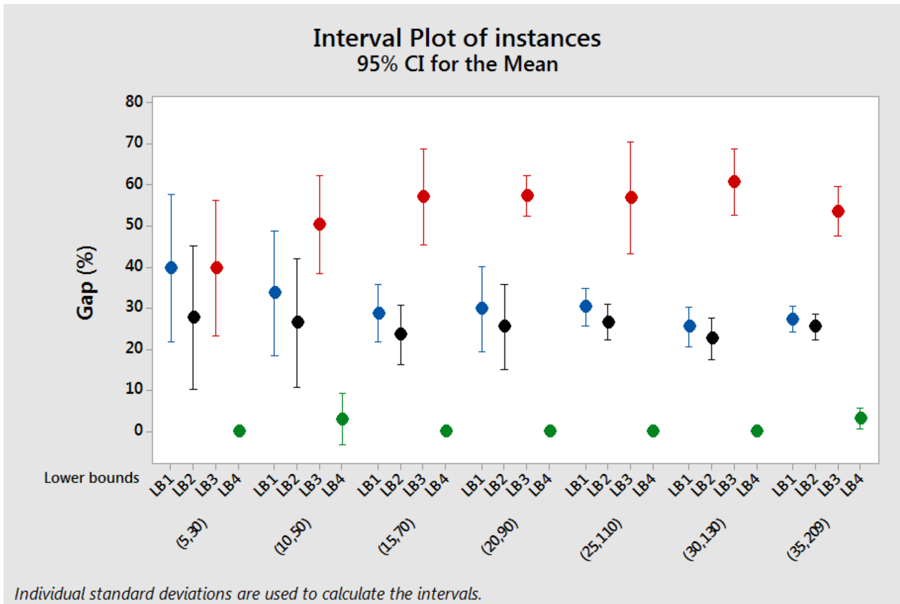

Figure 10: Performance comparison of the lower bounds with the obtained optimal solutions by CPLEX

the optimal solution. Because, for defining this lower bound the minimum number of constraints (assumptions) are relaxed comparing to the general problem. The ranking order for these lower bounds is as follows:

$Gap_{LB_4} < Gap_{LB_2} < Gap_{LB_1} < Gap_{LB_3}$

Moreover, an analysis of the variance (ANOVA) with a confidence level of 95% was taken using the Minitab.17 software to check the statistical validity of the results (Fig. 10). As can be seen in this figure, for each problem size the interval of the gaps for all the proposed lower bounds (LB_1, LB_2, LB_3, LB_4) are presented. In all the cases, LB_4 has the minimum interval of the gaps.

6. Conclusion

Three categories of the energy-efficient single machine scheduling problems, when the machine has several states, are addressed in this study. The complexity of the problems with the same energy price, increasing (decreasing) energy price during all the

Table 3: The comparison results between the proposed lower bounds and obtained optimal solutions by CPLEX in percentage

(n,T)		Gap_{LB_1}	Gap_{LB_2}	Gap_{LB_3}	Gap_{LB_4}
(5,30)	Min	19.41	8.14	18.94	0.00
	Average	**39.79**	**27.70**	**39.65**	**0.07**
	Max	54.75	40.68	56.00	0.37
(10,50)	Min	20.15	13.31	42.53	0.00
	Average	**33.66**	**26.44**	**50.27**	**2.90**
	Max	50.35	43.94	62.13	11.59
(15,70)	Min	20.02	14.51	47.44	0.00
	Average	**28.76**	**23.52**	**57.10**	**0.00**
	Max	35.60	30.88	68.39	0.00
(20,90)	Min	15.52	11.05	53.20	0.00
	Average	**29.83**	**25.43**	**57.38**	**0.00**
	Max	36.03	31.23	62.37	0.00
(25,110)	Min	25.96	22.44	44.55	0.00
	Average	**30.29**	**26.59**	**56.68**	**0.00**
	Max	34.27	30.65	70.64	0.00
(30,130)	Min	20.29	17.10	54.55	0.00
	Average	**25.45**	**22.51**	**60.67**	**0.00**
	Max	28.29	25.98	71.42	0.00
(35,209)	Min	24.34	22.57	48.42	0.00
	Average	**27.27**	**25.48**	**53.58**	**3.11**
	Max	30.68	28.91	61.46	5.17
Average		**30.72**	**25.38**	**53.62**	**0.87**

time-slots, and TOU energy price, with the objective of the total energy consumption costs minimization (TEC), are analyzed. First of all, we proved that when the jobs' sequence is fixed and TOU energy price is considered ($1,TOU|sequence,\ states|TEC$), unlike what the authors considered in [25], the problem is polynomial. Then, we also proved that for the case without the fixed sequence, when the energy prices are constant or increasing ($1,c_t = c|states|TEC$ and $1,c_t < c_{t+1}|states|TEC$), these problems are polynomial. But, for the problems with the TOU energy price, when the jobs have different processing times, the problem is NP-hard. Moreover, some lower bounds solution for the $1,TOU|states|TEC$ problem are presented.

In the future works, it could be interesting to analyze the complexity of other versions of this problem, i.e. when the jobs have different energy consumptions and the machine is able to process the jobs at different speeds. In addition, considering other assumptions such as the release dates and the due dates for each job the setup times for each state, and dealing with a more complex system like job shop, open shop and flow shop systems, with more than one machine can be established in future research.

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