Joint optimization of monitoring quality and replacement decisions in condition-based maintenance

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Khanh T.P Nguyen[a], Phuc Do[b], Khac Tuan Huynh[c], Christophe Bérenguer[d] and Antoine Grall[c]

[a]*Ecole Nationale d'Ingénieurs de Tarbes, INP Toulouse, LGP, France*
[b]*Univ. Lorraine,CRAN, UMR 7039, Vandoeuvre-les-Nancy, 54506, France*
[c]*Univ. Technology of Troyes, UMR 6281, ICD, ROSAS, LM2S, Troyes, France*
[d]*Univ. Grenoble Alpes, CNRS, Grenoble INP, GIPSA-lab, 38000 Grenoble, France*

Abstract

The quality of condition monitoring is an important factor affecting the effectiveness of a condition-based maintenance program. It depends closely on implemented inspection and instrument technologies, and eventually on investment costs, i.e., a more accurate condition monitoring information requires a more sophisticated inspection, hence a higher cost. While numerous works in the literature have considered problems related to condition monitoring quality, (e.g., imperfect inspection models, detection and localization techniques, etc.) few of them focus on adjusting condition monitoring quality for condition-based maintenance optimization. In this paper, we investigate how such an adjustment can help to reduce the total cost of a condition-based maintenance program. The condition monitoring quality is characterized by the observation noises on the system degradation level returned by an inspection. A dynamic condition-based maintenance and inspection policy adapted to such a observation information is proposed and formulated based on Partially Observable Markov Decision Processes. The use and advantages of the proposed joint inspection and maintenance model are numerically discussed and compared to several inspection-maintenance policies through numerical examples.

Keywords: Condition Monitoring quality, Condition Based Maintenance strategy, Gamma process, Maintenance optimization, Partially Observable Markov Decision Processes.

1. Introduction

Condition monitoring (CM) is an important part in a condition-based maintenance (CBM) program as it can provide useful information about the system state for maintenance decision-making to improve the durability, reliability, and maintainability of industrial systems [13, 28]. This leads to a steady growth of CBM optimization models in the literature which are more advanced and better adapted to practical

☆Corresponding author.
Email address: `nguyethipk85@gmail.com or tnguyen@enit.fr` (Khanh T.P Nguyen[a])

industrial concerns. The performance of CBM policies with periodic inspections for single-unit stochastic deteriorating systems has been investigated in [8, 12, 15]. Moreover, different studies aimed to optimize the time interval between two successive inspections have been presented in the literature. In fact, it becomes more interesting to adapt the inspection interval according to the observed level of degradation state [3, 4] or according to the residual useful life (RUL) of the system [5, 9, 30].

However, these above studies are based on the assumption of perfect condition monitoring which returns the real system state without errors. This assumption is not always verified in practical applications because, in spite of the progress of sensor technology and monitoring techniques, the CM data are most often corrupted by noise and disturbances. To deal with this problem, numerous works in the literature have been proposed. Newby and Barker in [20] considered an imperfect inspection model in which the observed deterioration state is subject to Gaussian error. Using Hidden Markov Model theory, Neves *et al.* studied in [19] how the model parameters estimated from imperfect observations can affect the optimization of CBM strategies. Ghasemi *et al.* in [10] developed a partially observed Markov decision process (POMDP) to optimize the maintenance policy for a system whose state is hidden and can be estimated based on CM data. A continuous-state POMDP coupled with a normalized unscented transform for non-linear action models was proposed in [27] to formulate the problem of decision-making for optimal management of civil structures. In [1], the authors presented an effective approach to solve MDP/POMDP problems for optimal sequential decision-making in complex, large scale, non-stationary, partially or fully observable stochastic engineering environments. For recent studies, the relation between the value of information and numerous key features of the monitoring system was investigated in [16]. In [18], the authors proposed a methodology for an integral risk-based optimization of inspections in structural systems. The optimization problem is formulated based on a heuristic approach that is based on periodic inspection campaigns, a fixed repair criterion and does not consider the inspection-quality-adjustment options.

In reality, the quality of CM depends closely on the inspection and instrumentation technologies implemented, and ultimately on the costs invested. Therefore, its quality level could be controlled by adjusting inspection costs, i. e. paying higher costs to implement better monitoring devices or to perform more thorough analysis of the deterioration, and to obtain more accurate CM information. The issue of choosing between several kinds of inspection, or monitoring tools, at different costs, in order to adjust the inspection quality for a better decision-making has been investigated in several works, e.g. [6, 7, 24, 25]. However, these works are mainly developed in a somehow different setting than the one considered in the present work: they consider that i) the system evolution follows intrinsically discrete states, and ii) the possible different inspections return the value of a discrete state, they have to be chosen within a finite predetermined set, and the observation probability matrix is fixed and known in advance for each of these inspections. In our setting, the system is basically subject to a continuous deterioration that is monitored by inspection, and this continuous-state deteriorating system with continuous observation is then mapped onto a discrete model for maintenance decision-making ; the observation matrices are thus estimated at each decision time and adapted to the actual deterioration and observation characteristics of the system. As for the case of continuous observations for a continuous degradation process, whose transition matrices can be obtained through simulation, it has been already studied in [14]. Inspired from these previous studies, the present work investigates the performance of a dynamic inspection-maintenance policy for a system subject to a continuous degradation process. The quality of the degradation information returned by inspections is characterized by the variance parameter of random errors following a Gaussian distribution, [20]. In [22], the authors developed a new flexible inspection strategy whose decision rules are adapted to this variance. It addresses the question of whether and when the adjustment of inspection quality from low level to high level is necessary, and underlines the value of CM quality adjustment in CBM optimization. This paper extends the work presented in [23], and develops two main original contributions: i) the proposition of a POMDP dynamic maintenance management framework based on continuous deterioration processes with imperfect monitoring, and ii) the in-depth performance assessment of the proposed framework and its in detail comparison with currently used CBM approaches.

- Regarding the first contribution, this work proposes a discretization formulation for a continuous degradation process with random observation noise, so that the POMDP decision framework can be deployed and implemented starting from the continuous deterioration characteristics of the considered system. This approach allows to relax the requirement that the conditional probability of the discrete observation given the system state is known and to connect more tightly the upper-level maintenance decision process with the physical deterioration of the maintained system. In addition, the imperfect inspection quality characterized and modeled by an additive observation noises is investigated and the resulting integrated imperfect inspection model, taking into account jointly the quality and the cost of an inspection, is also discussed. A comprehensive cost model including maintenance and inspection costs is then developed to evaluate the performance of the proposed joint CBM maintenance and monitoring policy.

- As for the second contribution, the behavior and the performance of the proposed joint policy are numerically assessed and analyzed, and compared to currently used inspection and maintenance policies. Sensitivity analyses regarding the performance of the proposed policy are also investigated and discussed. Finally, the use and the advantages of the proposed models are illustrated and highlighted.

The remainder of this paper is organized as follows. Section 2 describes the problem statement and its mathematical formulation. Section 3 presents the proposed dynamic inspection-maintenance policy. In addition, cost models and optimization processes are also discussed and formulated. Numerical experiments are presented in Section 4. The proposed inspection-maintenance policy is herein numerically analyzed.

Three variants of the proposed policy are also discussed to examine the performance of proposed general policy. Finally, conclusions and future research directions are summarized in Section 5.

2. Problem statement and mathematical formulation

2.1. System description and assumptions

Consider a single unit system subject to a stochastic continuous degradation process $\{X_t\}_{t\geq 0}$, $X_t \in \mathbb{R}^+$ that evolves monotonically from the new state to the failed state in the absence of maintenance actions. The system fails when its degradation level exceeds a fixed failure threshold L, $X_t \geq L$. The failure state is recognized without any inspection (i.e., self-announcing failure). Let denote respectively F_{X_t} and f_{X_t} the cdf and the pdf of the degradation process $\{X_t\}$ at time t.

The system degradation is often hidden, monitoring is then required to reveal the degradation level. "Continuous" monitoring, i.e. performed at each time step (Δt), is usually very costly, and may be impossible to implement in some specific practical engineering applications [21]. Note that Δt is nothing but the minimum time period at which the system can be accessed (and at which it could make sense to access it) for inspection. In practice, the value of Δt depends on both the monitoring system characteristics and the time behavior of the monitored system. Depending on these, Δt can range form seconds (for fast evolving systems) to e.g. years... (for slowly deteriorating systems). To make our modeling framework independent of this time scale, we take Δt equal to 1 (in arbitrary time units). In this framework, it is more suitable to implement periodic inspection whose length between two successive inspections T is a multiple of Δt [2]. Then, at the beginning of each observation period, if the system has failed, it is immediately replaced by a new one. Otherwise, an inspection is carried-out to reveal the system state (degradation level) and then based on the obtained information, the preventive maintenance decision can be made. However, from a practical point of view, inspection operations may not reveal exactly the true system degradation state because of noise or poor measurements. Accordingly, it is assumed that at each inspection time T_n, the observed state of the system, denoted Y_{T_n}, can be described as

$$Y_{T_n} = X_{T_n} + \epsilon_q, \tag{1}$$

where,

- X_{T_n} is the true degradation level of the system at time T_n;
- ϵ_q is the measurement error and can be described by a random variable;
- q indicates the quality index of an inspection action.

It is assumed that the measurement errors are described by a Gaussian distribution $\mathcal{N}\left(0,\sigma_q^2\right)$ with probability density function

$$G_{\sigma_q}\left(x\right)=\frac{1}{\sigma_q\sqrt{2\pi}}\mathrm{e}^{-\frac{1}{2}\left(\frac{x}{\sigma_q}\right)^2}. \tag{2}$$

The standard deviation σ_q represents the inspection quality, i.e. an inspection with higher quality returns smaller variance of noise [20]. In that way, several inspection quality levels are herein investigated, e.g., $\epsilon_q = 0$ for a perfect inspection action. That means the observation exactly reveals the hidden system state. Note also that the inspection quality is usually increasing with the inspection cost, a quality-based inspection model will be described in Section 3.2.

2.2. Problem statement of inspection quality adjustment and replacement decisions: an adaptive POMDP-based decision model

For inspection and maintenance decision-making, the crucial questions raising here are (i) whether or not investing in the improvement of inspection quality to optimize the total maintenance cost, and (ii) how to adapt the maintenance decision to a given inspection quality. To answer to these questions, in this paper, a POMDP-based inspection and maintenance model is proposed to optimize the total maintenance cost over a planning horizon $[0, T_{end}]$.

The POMDP framework has been widely used to model a sequential decision process in which the system dynamics are characterized by a Markov Decision Process, but whose underlying states cannot be directly observed. In detail, the POMDP is defined in discrete time and formally determined by a 7-tuple ($\mathbf{S}_Z$, $\mathbf{A}$, $\mathbf{P}_T$, $\mathbf{C}$, $\mathbf{S}_O$, $\mathbf{P}_O$, γ), in which:

- $\mathbf{S}_Z$ and $\mathbf{S}_O$ are respectively the sets of system discrete states and discrete observations. In order to apply the POMDP model for a single unit system subject to a stochastic continuous degradation process with Gaussian observation errors, it is necessary to discretize the system states and also their observation states. This step is presented in detail in Subsection 2.3. The probabilistic state transition law and conditional observation law are also derived in this section.
- $\mathbf{A}$ is the set of actions. For our problem, the set of actions $\mathbf{A}$ consists of two subsets that are the inspection quality level options (q) and the maintenance options (Replace (R) or Do-nothing (DN)).
- $\mathbf{P}_T$ is the set of conditional transition probabilities between states. It is derived in Subsection 2.3.1.
- $\mathbf{C}: \mathbf{S}_Z \times \mathbf{A} \to \mathbb{R}$ is the cost function that is connected to the actions and the system states. Its formulation is presented in Subsections 3.2 and 3.3.
- $\mathbf{P}_O$ is the set of conditional observation probabilities. In Subsection 2.3.2, the conditional observation probabilities are derived.

- $\gamma \in [0, 1]$ is the discount factor. In this paper, we are only interested in the expected sum of future cost and do not consider the discounted value, so $\gamma = 1$.

Figure 1 illustrates the POMDP-based inspection and maintenance process. At the beginning of the decision period T_n, if the system still works, the appropriate actions for system are investigated. Recall that the underlying discrete system state, noted Z_{T_n}, is hidden: we only have the prior information about the probability distribution of the current state, called belief function b_{T_n}, that is derived from the last period. To update the belief function at this moment, it is necessary to decide the inspection quality level (q) to perform an inspection. The details of the decision optimization (ie. how to choose an appropriate inspection) are presented in Section 3. Given an observation O_{T_n}, the belief function is updated, see Subsection 2.4 for details. Then, the appropriate maintenance action is decided, see Section 3 for the details of the maintenance policy. If the DN option is chosen, the system state does not change. Contrarily, when the R option is chosen, the system is restored to its new state. Then, the corresponding belief function is derived for the next period, see Subsection 2.4 for the details of its transition process.

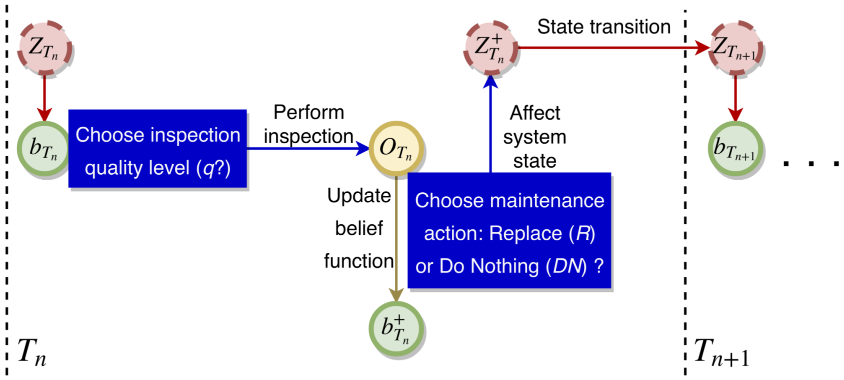


Figure 1: Illustration of POMDP-based inspection and maintenance process

2.3. State discretization modeling and formulation

The first step towards the development of a POMDP-based modeling approach for maintenance evaluation purpose is the discretization of the continuous-time continuous-state true deterioration process $\{X_t\}_{t\geq 0}$ into continuous-time discrete-state process $\{Z_t\}_{t\geq 0}$. This discretization step is not only required from a methodological point of view for the model development, it is also interesting from a practical point of view. Indeed, very often in practice, even in the case of continuously deteriorating items, the maintenance decision-maker considers only a few discrete deterioration states, such as : "good", "minor deterioration", "medium deterioration", "severe deterioration", "critical deterioration", "failed". From a practical point of view, for the decision-maker, considering only a few discrete deterioration states allows having a more synthetic view on the system state, and allows a simpler maintenance decision-making process [29]. In order to comply with this observed practice, in our proposed setting, we consider the number of discrete deterioration states N as an input for our modeling approach and for the policy optimisation, and not as a decision variable to be optimized.

The discrete-state space of $\{Z_t\}_{t\geq 0}$ is then defined by $\mathbf{S}_Z = \{1, 2, \ldots, N, N+1\}$, where the state $N+1$ is the system failure state that corresponds to the interval $[L, +\infty)$ for the degradation level, and a state $k \in \{1, 2, \ldots, N\}$ is a degradation state that corresponds to the interval $[(k-1)\,l, kl)$, where $l = \frac{L}{N}$.

The observed deterioration process $\{Y_{T_n}\}_{n\in\mathbb{N}}$, $Y_{T_n} \in \mathbb{R}$, is then a discrete-time continuous-state stochastic process. Note that in practice the state space of $\{Y_{T_n}\}_{n\in\mathbb{N}}$ should be $\mathbb{R}^+$, the theoretical state space $\mathbb{R}$ approaches the practical one when σ_q is not too large. As $\{X_t\}_{t\geq 0}$, $\{Y_{T_n}\}_{n\in\mathbb{N}}$ is also discretized in $N+1$ state as $\mathbf{S}_O = \{1, 2, \ldots, N, N+1\}$ to obtain the discrete-state observed deterioration process $\{O_n\}_{n\in\mathbb{N}}$, where the states 1 and $N+1$ correspond to the intervals $(-\infty, l)$ and $[L, +\infty)$ respectively, and a state $h \in \{2, \ldots, N\}$ corresponds to the interval $[(h-1)\,l, hl)$ with $l = \frac{L}{N}$. The process $\{O_{T_n}\}_{n\in\mathbb{N}}$ is thus a discrete-time discrete-state stochastic process. At an inspection time $t = T_n$, given the true deterioration level $Z_{T_n} = k$, $k \in \{1, \ldots, N+1\}$, if the observed state $O_n = k$, then the system state is correctly detected; otherwise (i.e., $O_n = h \neq k$), the detection is wrong. Figure 2 shows an illustration of the system deterioration modeling and discretization approach.

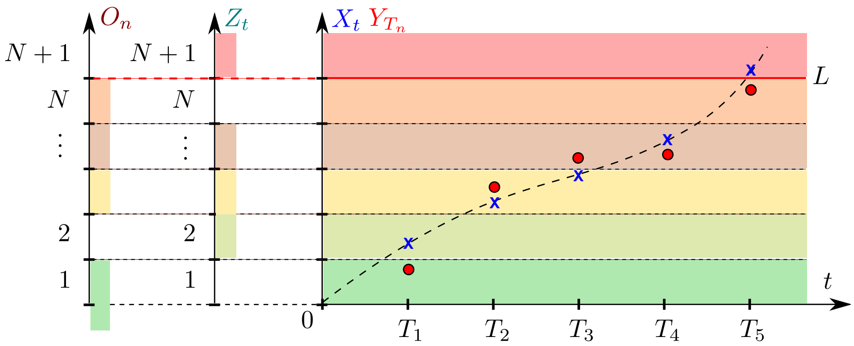


Figure 2: Illustration of degradation-based failure model and discretization approach

2.3.1. Derivation of the state transition law $P\left(Z_{T_{n+1}} = m \mid Z_{T_n} = k\right)$

We derive in this section the expression of the state transition law $P\left(Z_{T_{n+1}} = m \mid Z_{T_n} = k\right)$, which is the conditional probability that the system is in the discrete state m at the inspection date T_{n+1} given that it is in the discrete state k at the inspection date T_n, where $k, m \in \mathbf{S}_Z$ and $m \geq k$. Recall that $F_{X_{T_n}}$ and $f_{X_{T_n}}$ are respectively the cdf and the pdf of the degradation process, the four following configurations of m and k are considered.

- If $1 \leq k = m \leq N$,

$$P\left(Z_{T_{n+1}} = k \mid Z_{T_n} = k\right) = \frac{\int_{(k-1)l}^{kl} P\left(X_{T_{n+1}} - X_{T_n} < kl - x\right) f_{X_{T_n}}(x)\, dx}{F_{X_{T_n}}(kl) - F_{X_{T_n}}((k-1)\,l)}, \tag{3}$$

- If $1 \leq k < m \leq N$,

$$P\left(Z_{T_{n+1}} = m \mid Z_{T_n} = k\right) = \frac{\int_{(k-1)l}^{kl} P\left((m-1)\,l - x \leq X_{T_{n+1}} - X_{T_n} < ml - x\right) f_{X_{T_n}}(x)\,dx}{F_{X_{T_n}}(kl) - F_{X_{T_n}}((k-1)\,l)}, \quad (4)$$

- If $1 \leq k \leq N$, $m = N+1$,

$$P\left(Z_{T_{n+1}} = N+1 \mid Z_{T_n} = k\right) = \frac{\int_{(k-1)l}^{kl} P\left(X_{T_{n+1}} - X_{T_n} \geq L - x\right) f_{X_{T_n}}(x)\,dx}{F_{X_{T_n}}(kl) - F_{X_{T_n}}((k-1)\,l)}, \quad (5)$$

- If $k = N+1$, $m = N+1$,

$$P\left(Z_{T_{n+1}} = N+1 \mid Z_{T_n} = N+1\right) = P\left(L \leq X_{T_{n+1}} \mid L \leq X_{T_n}\right) = 1. \quad (6)$$

The expressions $P\left(Z_{T_{n+1}} = m \mid Z_{T_n} = k\right)$ in the special case of a Gamma deterioration process are detailed in Appendix A.

2.3.2. Derivation of the conditional observation probability $P_q\left(O_n = h \mid Z_{T_n} = k\right)$

We derive in this section the expression of the conditional observation probability $P_q\left(O_n = h \mid Z_{T_n} = k\right)$ which is the conditional probability that the discrete observation is h given that the system is in the discrete state k (where $h \in \mathbf{S}_O$ and $k \in \mathbf{S}_Z$), under an inspection quality index q. Recall that G_{σ_q} is the cdf of the Gaussian noise and that $F_{X_{T_n}}$ and $f_{X_{T_n}}$ are respectively the cumulative distribution function (cdf) and the probability density function (pdf) of the degradation process. To derive the expression of the conditional observation probability, the following cases have to be distinguished:

- If $h \in [2, N]$ and $k \in [1, N]$,

$$P_q\left(O_n = h \mid Z_{T_n} = k\right) = \frac{\int_{(k-1)l}^{kl} \left(G_{\sigma_q}(hl - x) - G_{\sigma_q}((h-1)\,l - x)\right) f_{X_{T_n}}(x)\,dx}{F_{X_{T_n}}(kl) - F_{X_{T_n}}((k-1)\,l)}, \quad (7)$$

- If $h = N+1$ and $k \in [1, N]$,

$$P_q\left(O_n = N+1 \mid Z_{T_n} = k\right) = \frac{\int_{(k-1)l}^{kl} \left(1 - G_{\sigma_q}(L - x)\right) f_{X_{T_n}}(x)\,dx}{F_{X_{T_n}}(kl) - F_{X_{T_n}}((k-1)\,l)}, \quad (8)$$

- If $h \in [2, N]$ and $k = N+1$,

$$P_q\left(O_n = h \mid Z_{T_n} = N+1\right) = \frac{\int_{L}^{\infty} \left(G_{\sigma_q}(hl - x) - G_{\sigma_q}((h-1)\,l - x)\right) f_{X_{T_n}}(x)\,dx}{1 - F_{X_{T_n}}(L)}, \quad (9)$$

- If $h = N+1$ and $k = N+1$,

$$P_q\left(O_n = N+1 \mid Z_{T_n} = N+1\right) = \frac{\int_{L}^{\infty} \left(1 - G_{\sigma_q}(L - x)\right) f_{X_{T_n}}(x)\,dx}{1 - F_{X_{T_n}}(L)}, \quad (10)$$

- If $h = 1$ and $k \in [1, N+1]$,

$$P_q\left(O_n = 1 \mid Z_{T_n} = k\right) = 1 - \sum_{h=2}^{N+1} P_q\left(O_n = h \mid Z_{T_n} = k\right). \tag{11}$$

These different expressions of $P\left(O_n = h \mid Z_{T_n} = k\right)$ are further detailed in Appendix B for the special case of a Gamma deterioration process.

The integrals in Eqs.(3-11) are numerically evaluated using the Gauss-Kronrod quadrature formula [11] implemented in the `integrate` R function.

2.4. Belief function

Since the real degradation state of system cannot be revealed exactly by imperfect inspections, then the state of knowledge of the decision-maker on the system state at time t is characterized by a belief function b_t consisting of a vector of the probabilities of the real system degradation level over the discrete-state space $\mathbf{S}_Z$. Each element of b_t is defined as $P(Z_t = k)$, $k \in \mathbf{S}_Z$. For a new system, the initial belief function, noted b_0 is known without inspection: $b_0 = [1, 0, 0...0]$, i.e. the probability of the new state is equal to 1 and the one of other states is 0.

As the observation measure obtained after an inspection at T_n depends on the system state and the inspection quality index q, then $P_q(O_n = h)$ the probability that the observation measure is h is given by:

$$P_q(O_n = h) = \sum_{k \in \mathbf{S}_Z} P(Z_{T_n} = k) \cdot P_q(O_n = h | Z_{T_n} = k) \tag{12}$$

Given the observation measure $O_n = h$ after an inspection with quality index q, we update the belief function $b_{T_n^+}$ whose each element is given by:

$$P(Z_{T_n}^+ = k) = P(Z_{T_n} = k | O_n = h) = \frac{P(Z_{T_n} = k) \cdot P_q(O_n = h | Z_{T_n} = k)}{P_q(O_n = h)} \tag{13}$$

where $P(Z_{T_n}^+ = k)$ is the probability that the real degradation state is k given that the observation h is obtained after an inspection at quality index q. The value of the belief function at the next inspection period, without maintenance can then be evaluated as :

$$b_{T_{n+1}} = b_{T_n}^+ \cdot \mathbb{P}_{T_n}^{(z'|z)} \tag{14}$$

where $\mathbb{P}_{T_n}^{(z'|z)}$ is the state transition matrix of the discrete system state without maintenance action. Each element of the state transition matrix, that is the conditional probability that the system is in the discrete state m at the inspection date T_{n+1} given that it is in the discrete state k at the inspection date T_n, where $k, m \in \mathbf{S}_Z$ and $m \geq k$, have been derived in Subsection 2.3.1.

3. Dynamic inspection-maintenance policy

In this section, a dynamic inspection-maintenance policy is proposed. Within this policy, the maintenance decision for both inspection quality adjustment and preventive maintenance action is based on the knowledge on the system deterioration state summarized in the belief function vector. The maintenance cost is herein used as a criterion for the optimization process.

3.1. Policy description

At each periodic discrete time $T_n = T_{n-1} + T$ (T is a decision variable which needs to be optimized), if the system is still functioning, the online adaption process for the inspection quality and the maintenance decision structure are as follows:

- First, the belief function is derived from the information at the last period (see again Section 2.4). Then, an inspection quality index q is selected by minimizing the expected total maintenance cost corresponding to the belief function value at this moment. It should be noticed that the relationship between the belief function and the expected maintenance cost is discussed in Section 3.3. An associated inspection cost c_i^q is incurred when performing an inspection with quality index q;
- Given the deterioration state observation returned by the inspection operation, the belief function is then updated;
- Based on the updated belief function after inspection, the preventive maintenance action (replace the system (R) or do nothing (DN)) is selected by minimizing the expected total maintenance cost.

Figure 3 illustrates the decision process for inspection and maintenance decision-making.

The proposed policy involves two kinds of decision variables:

- The inspection period T: this is a global decision variable in the sense that it is set of the whole planning horizon.
- Two local decision variables, whose value is set for each discrete time period ($T_n = T_{n-1} + T$): inspection quality index q and maintenance action (replacement R or DN).

To find the optimal value of these decision variables, cost models are herein developed and presented in next sections.

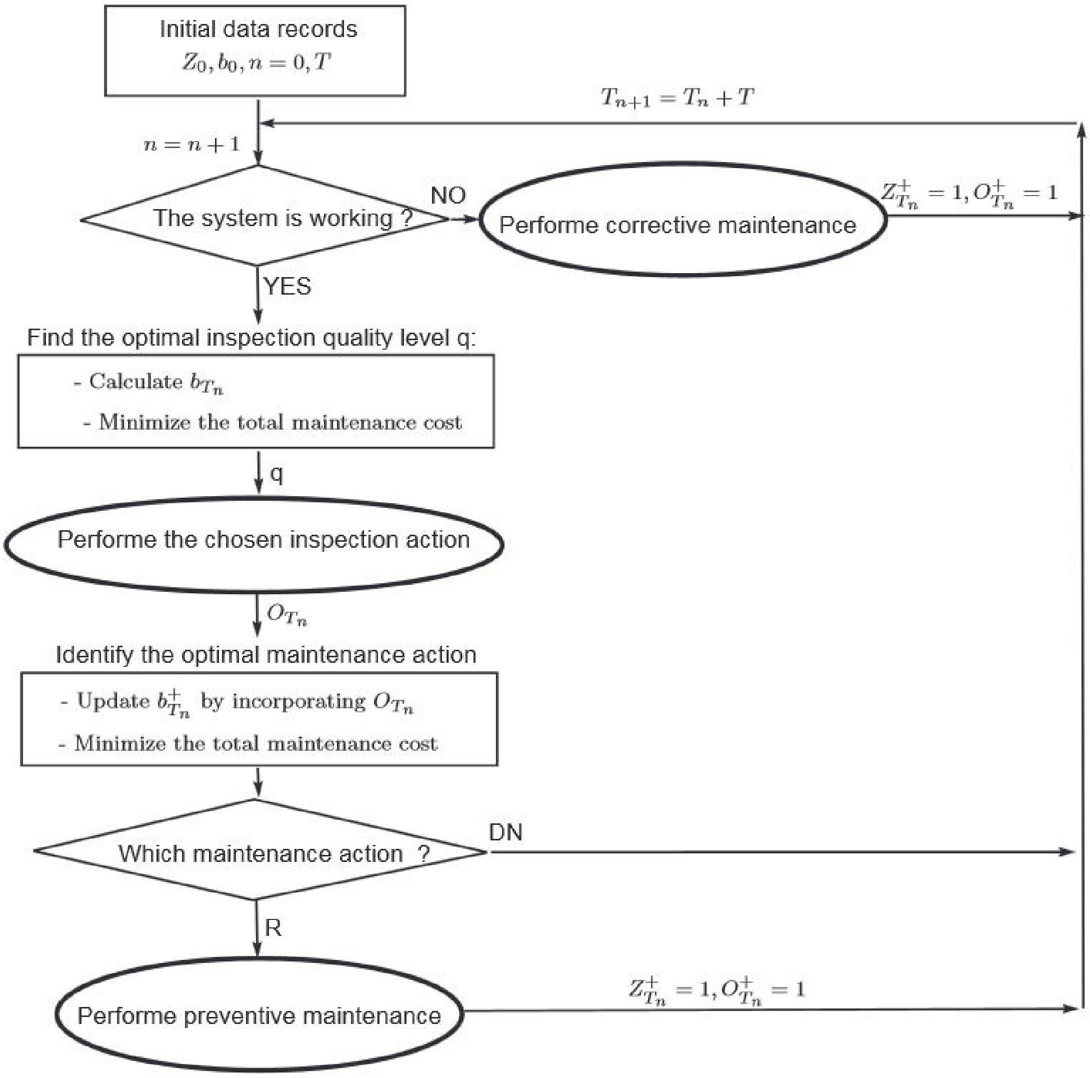


Figure 3: Illustration of the proposed maintenance policy.

3.2. Inspection cost formulation

It is pointed out in the literature that a higher quality inspection incurs a higher inspection cost, see for instance [5]. Therefore, we consider σ_q as a decreasing function of the corresponding inspection cost c_i^q. In that sense, it is assumed in this work that when the quality index q is chosen at inspection time T_n, one has to pay an inspection cost which is defined as:

$$c_i^q = c_i^l \cdot (\nu - (\frac{\sigma_q}{\sigma_l})^k \cdot (\nu - 1)), \tag{15}$$

where

- σ_l and c_i^l are respectively the standard deviation of the measurement errors and the inspection cost at the lowest inspection quality;

- ν is the ratio between the cost of the best quality inspection and the cost of the lowest quality inspection ($\nu = c_i^h / c_i^l$);
- k is the parameter characterizing the shape of the inspection cost function.

Note that, for the best quality inspection, we suppose that the observation reveals the real system state, $\sigma_q = 0$.

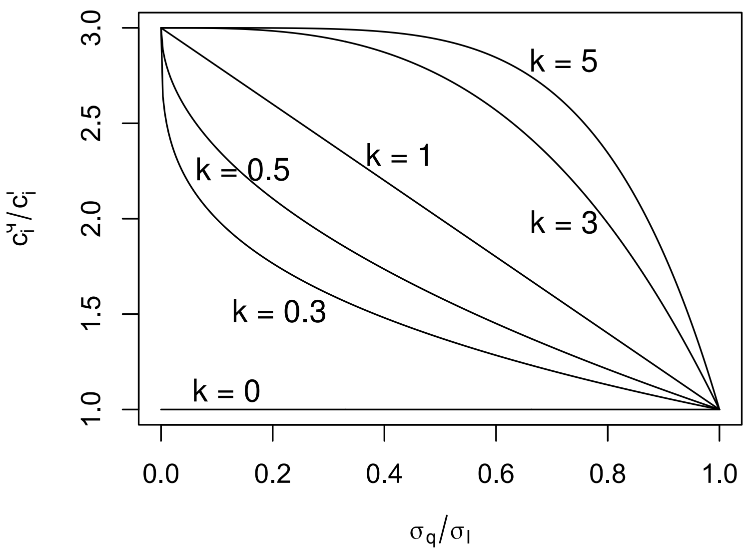


Figure 4: Illustration of inspection cost function, $\nu = 3$

According to this cost model, different shapes of the inspection cost function can be found depending on the value of k, see Figure 4 as an illustration:

- $k = 0$, the inspection cost is constant ($c_i^q = c_i^l$)
- $0 < k < 1$, the inspection cost is a convex function: the inspection cost decreases more than the decrease gain of inspection quality
- $k = 1$, the inspection cost is a linear function
- $k > 1$, the inspection cost is a concave function: the inspection cost decreases less than the decrease gain of inspection quality.

3.3. Maintenance total cost formulation and optimization

The optimal cost incurred by this policy is given by the minimal value of the expected cost $V_{[0,T_{end}]}(T, b_0)$ associated to different values of inspection period length T : $V_0^P = \min_T \left(V_{[0,T_{end}]}(T, b_0)\right)$, where $V_{[0,T_{end}]}(T, b_0)$ is calculated by :

$$V_{[0,T_{end}]}(T, b_0) = \mathbb{E}[C_D(b_0)] + V_{[T_1,T_{end}]}(T, b_{T_1}) \qquad (16)$$

At the initial time, the system is new, then we do nothing. Therefore, the system continuously operates until the first inspection at time T_1, the belief function at this time is known as b_{T_1} (calculated by Eq.((14)). Hence, the expected accumulated cost $V_{[0,T_{end}]}(T, b_0)$ over the planning period $[0, T_{end}]$ is the sum of the expected downtime cost $\mathbb{E}[C_D(b_0)]$ during period $[0, T_1]$ and the expected accumulated cost from the observation period T_1 until the final time T_{end}.

A replacement, whether preventive or corrective, can only be instantaneously performed at inspection times. Therefore, there exists the possibility of system failure, and an additional cost is incurred from the failure time until the next replacement time with down-time cost rate, c_d. Let $\mathbb{E}[C_D(Z_{T_n} = k)]$ be the expected downtime cost of the system during the period $[T_n, T_{n+1}]$ when knowing the real state $Z_{T_n} = k$ at T_n, then the expected downtime cost $\mathbb{E}[C_D(\cdot)]$ of the system during the period $[T_n, T_{n+1}]$ when knowing the belief function b_{T_n} is given by:

$$\mathbb{E}[C_D(b_{T_n})] = \sum_{k \in \mathbb{S}_z} P(Z_{T_n} = k) \cdot \mathbb{E}[C_D(Z_{T_n} = k)] \tag{17}$$

where the evaluation of $\mathbb{E}[C_D(Z_{T_n} = k)]$ is made using a discretization method. Recall that the length between two successive inspections T is a multiple of Δt: $T = J \cdot \Delta t$, in order words, the interval $[T_n, T_{n+1}]$ can be discretized into J sub-interval of length Δt, that is $[T_n, T_n + \Delta t[$, $[T_n + \Delta t, T_n + 2\Delta t[$, ..., and $[T_n + (J-1)\Delta t, T_{n+1}[$. If the failure time belongs to one of these intervals, we approximate it by the left end value. Given c_d the downtime cost rate c_d , if the failure time belongs $[T_n + (j-1)\Delta t, T_n + j\Delta t[$, where $(1 \leq j \leq J,\ j \in \mathbb{N})$ then the downtime cost associated is evaluated to $c_d \cdot (T_i - (j-1)\Delta t)$.

Hence, the expected downtime cost $\mathbb{E}[C_D(Z_{T_n} = k)]$ during interval $[T_n, T_{n+1}]$ is given by:

$$\mathbb{E}[C_D(Z_{T_n} = k)] = c_d \sum_{j=1}^{J} \left(P(Z_{T_n + j\Delta t} = N+1 | Z_{T_n} = k) - P(Z_{T_n + (j-1)\Delta t} = N+1 | Z_{T_n} = k\right) (T_i - (j-1)\Delta t) \tag{18}$$

where $P(Z_{T_n + j\Delta t} = N+1 | Z_{T_n} = k)$ can be evaluated be Eq.(5)

Let $\lfloor e \rfloor$ be the maximal integer that is inferior than e, then we define $N = \lfloor \frac{T_{end}}{T_i} \rfloor$ as the number of observation periods during $[0, T_{end}]$.

For $1 \leq n \leq N-1$, the expected accumulated cost from T_n until the final time T_{end} is given by:

$$V_n(T, b_{T_n}) = \mathbb{P}_f(T_n) C_C(\cdot) + (1 - \mathbb{P}_f(T_n)) \cdot \min_q \left(c_i^q + \sum_{h \in \mathbf{S}_O} P_q(O_n = h) \cdot \min\left[C_R(\cdot), C_{DN}(\cdot)\right] \right) \tag{19}$$

where $\mathbb{P}_f(t)$ is the probability that the system is failed at t. In detail, the equation Eq.(19) is evaluated by the sum of :

1. $C_C(b_{T_n}, T)$: the expected accumulated cost associated to system failure at the beginning of the observation period T_n, see Eq.(20). $C_C(b_{T_n}, T)$ is the sum between the corrective replacement cost c_c, the

expected downtime cost $\mathbb{E}[C_D(b_{T_n})]$ during $[T_n, T_{n+1}]$, and the expected accumulated cost from the next period until the end of the planning horizon $V_{n+1}(T, b_{T_1})$.

2. The expected accumulated cost associated to the case where the system still works at the beginning of the observation period T_n. In this case, it is necessary to decide the quality level for inspection. Then, based on the updated belief function, the adequate maintenance action is performed (preventive replacement (R) or do nothing (DN)). The quality index q is chosen such that the expected cost associated to this decision (accumulated from this time until the end of the planning horizon) is minimal. It is evaluated using the inspection cost c_i^q, the minimal value of the expected accumulated cost $C_R(\cdot)$ corresponding to the preventive replacement decision (see Eq.(21)) and the expected accumulated cost $C_{DN}(\cdot)$ corresponding to the do-nothing decision (see Eq.(22)).

   - $C_R(\cdot)$ is evaluated as the sum of the preventive replacement cost c_p, the expected downtime cost $\mathbb{E}[C_D(b_0)]$, and the expected accumulated cost from the next period until the end of the planning horizon $V_{n+1}(T, b_{T_1})$ (see Eq.(21)).
   - $C_{DN}(\cdot)$ is evaluated as the sum of the expected downtime cost $\mathbb{E}[C_D(b_{T_{n^+}})]$ during $[T_n, T_{n+1})$ and the expected accumulated cost from the next period until the end of the planning horizon $V_{n+1}(T, b_{T_{n+1}})$ (see Eq.(22)).

$$C_C(\cdot) = c_c + V_{n+1}(T, b_{T_{n+1}}) + \mathbb{E}[C_D(b_0)] \tag{20}$$

$$C_R(\cdot) = c_p + V_{n+1}(T, b_{T_{n+1}}) + \mathbb{E}[C_D(b_0)] \tag{21}$$

$$C_{DN}(\cdot) = V_{n+1}(T, b_{T_{n+1}}) + \mathbb{E}[C_D(b_{T_n^+})] \tag{22}$$

For the terminal condition of the planning horizon, we suppose that $V_{T_{end}} = 0$.

4. Numerical experiments

In this section, we present the results of a numerical experiment implemented to get a better insight into the behavior of the proposed inspection and maintenance policy. Through this example, the aim is to show how the proposed maintenance decision rule helps to choose whether and when it is necessary to improve the inspection quality, and to illustrate the benefit of implementing a flexible/dynamic inspection and maintenance policy instead of a static one.

For this example, the system degradation is assumed to follow a homogeneous Gamma process, see Appendix A. The parameters (characteristics and costs of the system deterioration, inspection and maintenance actions) used for the numerical experiments are represented in Table 1.

Table 1: Parameters set for the numerical example.

λ	β	L	k	c_p	c_c	c_d	T_{end}	c_i^l	ν	σ_l	σ_h
1	1	5	1	50	100	25	30	1	3	0.67	0

In the remainder of this section, benchmark maintenance policies are first presented: they are used for performance comparison with the proposed maintenance policy. The behavior of the proposed maintenance policy, tuned at its optimum, is then analyzed and compared to the behavior of the benchmark policies. Finally, the benefit of using the proposed maintenance policy in terms of maintenance cost is investigated and compared to the cost performance of the benchmark policies.

4.1. Presentation of the benchmark policies for performance analysis and comparison

In order to study the performance of the proposed dynamic inspection-maintenance policy (presented in Section 3), namely policy P, three variant policies (namely P1, P2 and P3) are considered:

- **Policy P1 - Classic CBM (Condition Based Maintenance) policy.** System degradation states are periodically inspected after period length T. If the observed degradation state is greater than a preventive replacement threshold M, the system is then replaced by a new one. The inspection quality cannot be adjusted. Either a CBM with low quality inspections (noted PO_1^l) or a CBM with high quality inspections (noted PO_1^h) is considered in this experiment and applied throughout a planning horizon $[0, T_{end}]$.

- **Policy P2 - CBM with dynamic inspections policy.** Similar to Policy P1, a CBM policy is applied based on the observed degradation state. The inspection quality can be adjusted at each period. In detail, at the beginning of an inspection period, we decide the quality index q for an instantaneous inspection with a cost c_i^q (depending on the inspection quality q).

- **Policy P3 - Dynamic maintenance policy with fixed inspection quality.** Similar to Policy1, the inspection quality cannot be adjusted: either a dynamic replacement policy with low quality inspections (noted PO_3^l) or with high quality inspections (noted PO_3^h) is applied throughout a planning horizon $[0, T_{end}]$. Given an observation, the belief function is updated and then the relevant expected cost is evaluated for each option (DN or R). Based on the comparison between these costs, we decide to replace the system or not.

Table 2 reports a summary of the proposed policy (policy P) and the variant policies considered for comparison (policy P1, P2 and P3).

Table 2: Overview of the 4 policies

	Maintenance policy	Inspection policy	Decision variables
Policy P	Dynamic	Adjusted	T set throughout the planning horizon, q at the beginning of each inspection period and action (R or DN) corresponding to the updated belief function
Policy P1	Fixed	Fixed	(T, M) set throughout the planning horizon
Policy P2	Fixed	Adjusted	(T, M) set throughout the planning horizon and q at the beginning of each inspection period
Policy P3	Dynamic	Fixed	T set throughout the planning horizon and action (R or DN) corresponding to the updated belief function at every inspection period

The formulation of the cost model for the three policies P1, P2 and P3 can be adapted from the cost model developed for the proposed policy in Section 3.3. The detailed mathematical developments are presented in Appendix C. For each policy, the optimal expected accumulated cost over a planning horizon is evaluated using the classical backward induction algorithm and the grid-based algorithm [17]. For numerical examples, the system states and observations are discretized by 6 states. Among them, $Z_{T_n} = 6$ is the failure state. Then, the belief space is discretized by a set of vectors of 6 elements e_i. Each element of the belief function represents the probability of the corresponding state, and $\sum_{i=1}^{6} e_i = 1$. For each value of T, using the backward induction algorithm, the cost function corresponding to each possible maintenance action at T_{end} is computed for the set of points in the belief space. The optimal action at this moment is the one leading to the minimal cost. Next, using the optimal cost function at T_n, we derive the cost function corresponding to every action at T_{n-1} for the belief space and then find the optimal action. This procedure is iterated until $n = 1$, so that the optimal expected cost over the planning horizon is obtained.

Let $V_0^{P_1^l}$ and $V_0^{P_1^h}$ be respectively the minimal value of the optimal expected accumulated cost incurred by Policy P1 with either low quality inspection or high quality inspections:

$$V_0^{P_1} = \min(V_0^{P_1^l}, V_0^{P_1^h}) \tag{23}$$

Similarly, the optimal expected accumulated cost for Policy P3 is given by:

$$V_0^{P_3} = \min(V_0^{P_3^l}, V_0^{P_3^h}) \tag{24}$$

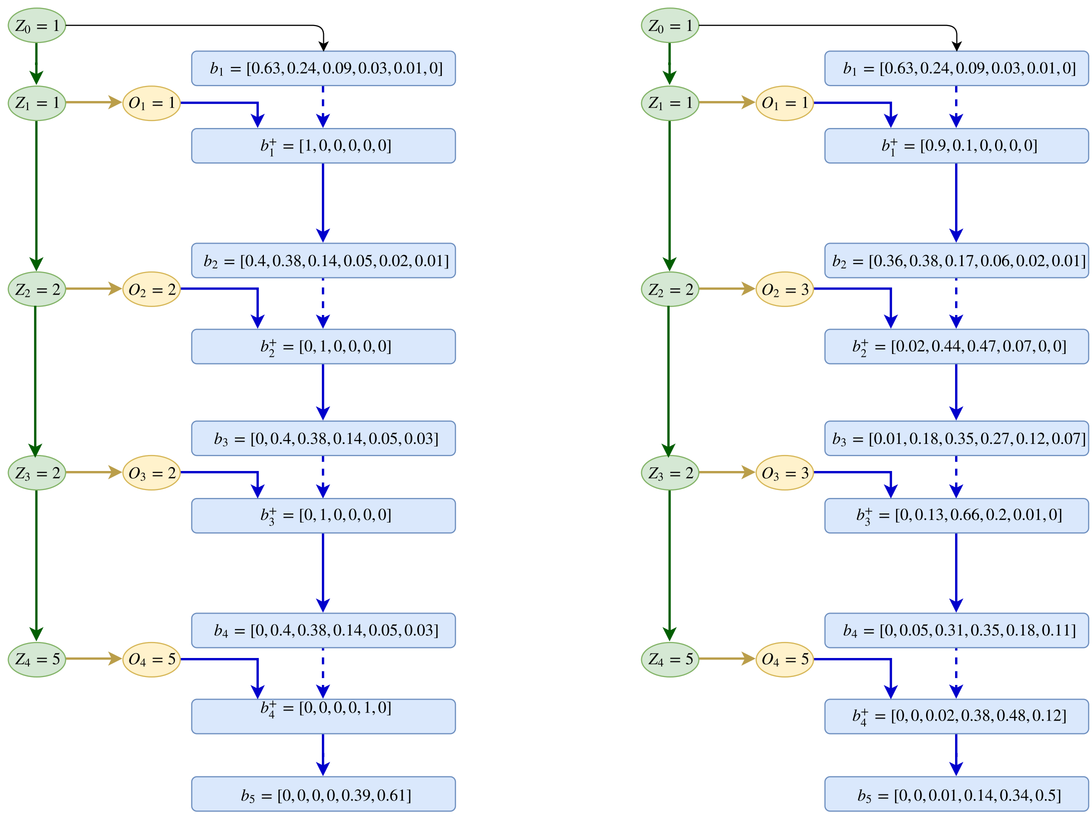
For policies with quality-adjusted-inspections (policies P and P2), six quality levels are examined. In details, the probability that an observation reveals the true system state is respectively $100\%, 90\%, 80\%, 70\%, 60\%$ and 50% for these six inspection levels. Note that q is the quality index, $q = 1$ characterizes the highest quality and $q = 6$ represents the lowest quality. At the initial moment, T_0, the system is totally new, therefore, its state is known and we are only interested in the maintenance options for next periods (from the the first inspection period, T_1).

4.2. Discussion of the belief function deviation under an adjusted-inspection-quality policy

The initial prior belief is assumed to be perfect. In practice, this assumption is reasonable because, considering a new system, it is trivial to have a perfect prior knowledge: $Z = 1$. In addition, at every stage, the belief function is updated according to the observation. Therefore, an adjusted-inspection-quality policy with the perfect inspection option allows correcting belief functions. Hence, the belief function does not derived so far from the truth. For an illustration, Figure 5 presents the changes of the belief function at the early stages of the policy application in two cases: highest quality inspection and lowest quality inspection. Note that, in Figure 5, for a simplification of the notations, the belief function b_{T_n} is represented by b_n. We consider a system whose the deterioration process is discretized by 6 states $Z = [1, 2, 3, ...6]$, with the failure state $Z = 6$. From the initial state $Z = 1$, after one decision period, the belief function is equal to $b_1 = [0.63, 0.24, 0.09, 0.03, 0.01, 0]$. Then, a highest quality inspection allows correcting the belief function thanks to its perfect outcome. Indeed, given the system state at the first stage, $Z_1 = 1$, the updated belief function in the case of the highest quality inspection, b_1^+ in sub-figure 6.(a), provides the probability that the system belongs to the initial state is 1, $P(Z_1 = 1) = 1$. In the case of the lowest inspection, see sub-figure 6.(b), thanks to the correct result, $O_1 = 1$, its relevant belief function indicates that the probability of the initial state is high $P(Z_1 = 1) = 0.9$. One can notice that even if using the lowest inspection, the belief function is close to the ground truth thanks to the correct inspection result. Next, after consecutive wrong observations, $O_2 = O_3 = 3$, the belief function deviates quite far from the truth, for example at the beginning of the fourth decision period, $b_4 = [0, 0.05, 0.31, 0.35, 0.18, 0.11]$, instead of the correct results $b_4 = [0, 0.4, 0.38, 0.14, 0.05, 0.03]$. However, thanks to the correct observation $O_4 = 5$, the belief function is updated and then, its value at the next period, i.e. $b_5 = [0, 0, 0.01, 0.14, 0.35, 0.5]$ does not deviate so far from the ground truth, i.e. $b_5 = [0, 0, 0, 0, 0.39, 0.61]$.

4.3. Analysis of the optimally tuned policies

In order to guarantee the solution accuracy, a large belief space is considered, consequently the computational cost is high. Using the MacBook Pro 3,1 GHz Intel Core i5, the computational time to optimize the policies 1, 2, 3 and 4 are respectively 980, 8550, 1020 and 16890 seconds. Considering the set of parameters presented in Table 1, all the considered policies are optimized and the following analysis can be made on their optimal structure:



(a) Highest quality inspection, $q = 1$

(b) Lowest quality inspection, $q = 6$

Figure 5: Illustration of belief function changes at the early stages.

- For maintenance policies with non-adjusted-quality-inspections (i.e. P1 and P3),
  - **Policy P1** - Under the optimal tuning of the decision variables, lowest quality inspections are performed at every inspection period $T = 1$ and the system is preventively replaced when the *observed* state is greater than 4, $O_{T_n} > 4$, which means that the replacement threshold M is 5. In summary, the optimal decision variables are $q^* = 6$ and $(T^*, M^*) = (1, 5)$.
  - **Policy P3** - Highest quality inspections ($q = 1$) are performed every period $T = 2$ and the replacement option triggered when the *degradation* state is equal or greater than 4, $Z_{T_n} \geq 4$. Note that for highest quality inspections, the observation reveals the true degradation state.

- For maintenance policies with adjusted-quality-inspections (i.e. P2 and P), the optimal action is decided by minimizing the relevant expected cost function that depends on the action and belief function. Figures 6 and 7 show the behavior of the policies at steady state (without the influence of the starting-ending conditions) ; note that the steady state behavior is guaranteed when there are still at least 11 periods from the current decision period to the end of the planning horizon, T_{end}.

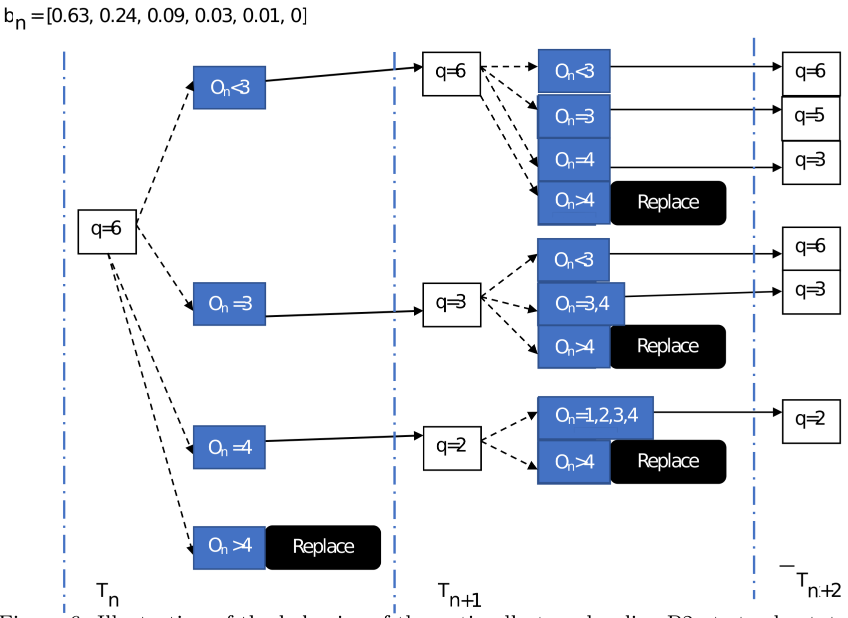


Figure 6: Illustration of the behavior of the optimally tuned policy P2 at steady state

As the belief function is strictly connected to the observation state and the historical process of the maintenance actions, in order to facilitate the description of the optimal policy, we present it in the format of the action process and the observation state. The behavior of these two policies, when optimally tuned, is sketched in Figures 6 and 7, and described below.

– **Policy P2** - The system is inspected at every period $T = 1$ and is preventively replaced when the *observed* state is greater than 4, i.e. $(T, M) = (1, 5)$. For an illustration, we assume that the belief function at the decision period T_n is $b_n = [0.63, 0.24, 0.09, 0.03, 0.01, 0]$. Then, the optimal stationary adjusted-inspection-quality policy in this case prescribes that at this inspection period, the lowest-quality-inspection $q = 6$ is implemented, see Figure 6. If the observation state O is lower than 3, the lowest-quality-inspection is still used at the inspection period T_{n+1}. In the case of the observation state $O = 3$ at T_n, the inspection with quality index $q = 3$ is performed at the next period, T_{n+1}, while for the higher observation result, $O = 4$, the system is inspected with higher inspection quality, $q = 2$. For $O > 4$, the system will be replaced by a new one, thus, a new maintenance cycle is then repeatedly performed. Next, at T_{n+1}, consider Figure 6, we find that the inspection quality is flexibly adjusted for every situation that depends on the historical action process and the observation gathered at T_{n+1}. Note that considering the planning horizon $T_{end} = 30$, the terminal condition impact is negligible for T_n, T_{n+1} and T_{n+2} when $T_{n+2} \leq 19$.

– **Policy P** - Under the optimally tuned policy P, the system is inspected at every period $T = 1$. For an illustration, we assume that the belief function at the decision period T_n is $b_n = [0.63, 0.24, 0.09, 0.03, 0.01, 0]$. Then, the optimal stationary policy in this case prescribes that at this inspection period, the lowest-quality-inspection $q = 6$ is implemented. Based on the updated

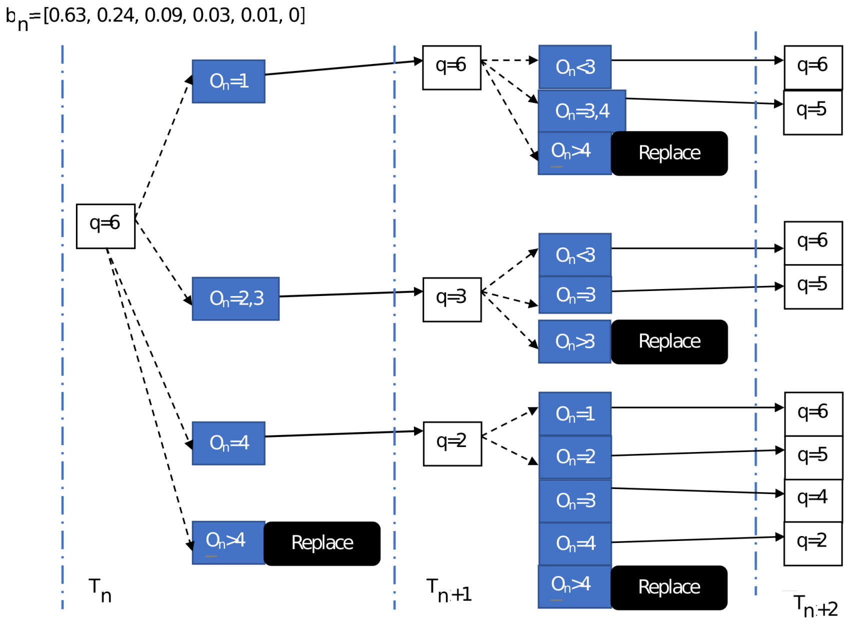


Figure 7: Illustration of the behavior of the Proposed Policy P at steady state, when optimally tuned

belief function corresponding to the observation state, the maintenance action is determined between DN ("Do-Nothing") or R ("Replace"). For example, see Figure 7, if the observed state at T_n is lower or equal to 4, $O \leq 4$, we do nothing (DN) while the system is replaced (R) when $O \geq 5$. In the case of the observation state $O = 1$ at T_n, without replacement the system is inspected with lowest quality ($q = 6$) at the next inspection period T_{n+1} and then is replaced for $O \geq 5$. If the observation state at T_n is $O = 2$ or 3, the inspection with quality $q = 3$ is implemented at T_{n+1} and the system is replaced for $O > 3$. Figure 7 shows a policy behavior under which both replacement option and the inspection quality are flexibly changed to adapt to every situation. Note that considering the planning horizon $T_{end} = 30$, the steady state is guaranteed for T_n, T_{n+1} and T_{n+2} when $T_{n+2} \leq 19$.

4.4. Comparison of the policies performance in terms of cost

In this section, we study and compare the cost performance of the considered policies. To this aim, let consider the percentage of the difference in their optimal values defined as follows:

$$\Delta_{12} = \frac{(V_0^{P_2} - V_0^{P_1})100\%}{V_0^{P_1}}; \qquad \Delta_{13} = \frac{(V_0^{P_3} - V_0^{P_1})100\%}{V_0^{P_1}}; \qquad \Delta_{14} = \frac{(V_0^{P} - V_0^{P_1})100\%}{V_0^{P_1}};$$
$$\Delta_{23} = \frac{(V_0^{P_3} - V_0^{P_2})100\%}{V_0^{P_2}}; \qquad \Delta_{24} = \frac{(V_0^{P} - V_0^{P_2})100\%}{V_0^{P_2}}; \qquad \Delta_{34} = \frac{(V_0^{P} - V_0^{P_3})100\%}{V_0^{P_3}}; \tag{25}$$

For example, Δ_{12} quantifies the relative cost increase (when $\Delta_{12} > 0$) or decrease (when $\Delta_{12} < 0$) between the optimal costs of Policy P2 and Policy P1. If $\Delta_{12} < 0$, Policy P2 is more cost efficient than Policy P1 since it incurs a lower cost over the considered planning horizon. The other Δ_{ij} ($i < j$, $1 \leq i \leq 3$ and $1 \leq i \leq 4$) can be interpreted in the same way.

4.4.1. Influence of the maintained system parameters on the relative performance of the maintenance policies

We investigate here the effect on the policies performance of the different characteristics of the maintained system (parameter characterizing the shape of the inspection cost function k, inspection cost c_i, downtime cost rate c_d, variance coefficient of the degradation process vc and the ratio between corrective and preventive replacement cost c_c/c_p). In order to compare the performance between the different inspection-maintenance policies, we consider only the difference (Δ_{ij}) in their optimal values. Table 3 presents the Pearson correlation values, that belong to $[-1, 1]$, between the system parameters and the relative cost difference Δ_{ij}. This Pearson correlation is equal to zero when there is no the correlation between two variables and equal to 1 (or -1) when two variables are positive (or negative) linearly proportional. The results presented in Table 3 are obtained from a set of numerical experiments with different values of $k = \{0.1, 0.3, 1, 3, 5\}$, $c_i \in [1, 5]$, $c_d \in [20, 35]$, $vc \in [0.1, 0.7]$, and $c_c/c_p \in [1, 5]$.

The following conclusions can be drawn:

- In general, the impact of the ratio between corrective and preventive maintenance cost c_c/c_p on the performance difference between these inspection-maintenance policies (Δ_{ij}) is not significant.

- The performance differences between adjusted-quality inspections and non-adjusted-quality inspections, that are characterized by the values of Δ_{12} and Δ_{34} , significantly depend on k, the parameter characterizing the shape of the inspection cost function.

- The performance differences between the dynamic replacement option and the fixed preventive replacement policy, that are characterized by the values of Δ_{13} and Δ_{24} , principally depend on c_i, the inspection cost and on vc, the variance coefficient of the degradation process.

- When considering the performance difference between Policy P2, the fixed preventive replacement policy with adjusted-quality inspections and Policy P3, the dynamic replacement option with non-adjusted-quality inspections, it is found that Δ_{23} depends strictly on k and might have a correlation with vc. These relations are further investigated in detail in subsection 4.4.3.

- The difference in performance between the proposed dynamic-inspection-replacement policy (Policy P) and the static one (Policy P1), that is characterized by Δ_{14}, does not depend neither on the downtime cost nor on the ratio between the preventive and corrective replacement cost.

After this general overview of the effects of the parameters of the maintained system on the relative performance of the different considered maintenance policies, the next sections further examine the sensitivity of the maintenance policy performance to the most influent parameters.

Table 3: Pearson correlation between the parameters inputs and the Δ_{ij}.
k: parameter characterizing the shape of the inspection cost function, c_i: inspection cost, c_d: downtime cost rate, vc: variance coefficient of degradation process, and c_c/c_p: the ratio between corrective and preventive replacement cost

	k	c_i	c_d	vc	c_c/c_p
Policy P1 vs P2 (Δ_{12})	0.68	0.57	0	-0.03	0
Policy P1 vs P3 (Δ_{13})	0	0.89	-0.14	0.88	-0.11
Policy P1 vs P (Δ_{14})	0.52	0.76	-0.1	-0.69	-0.04
Policy P2 vs P3 (Δ_{23})	-0.83	-0.07	-0.09	0.04	-0.01
Policy P2 vs P (Δ_{24})	-0.12	0.81	-0.27	0.69	-0.1
Policy P3 vs P (Δ_{34})	0.78	0.48	-0.05	0.04	0

4.4.2. Adjusted-quality vs. fixed-quality inspections: maintenance performance comparison

In this section, we investigate the performance gain brought by adjusted-quality inspections over non-adjusted ones. We firstly consider the policies with a fixed preventive replacement threshold (i.e. P1 and P2) and then the policies with dynamic replacement option (i.e. P and P3). In detail, corresponding to every combination of the input parameters (c_i^l, k, etc.) we find the minimum maintenance cost of 4 maintenance policies and then calculate the percentage of their difference, e.g. Δ_{12} and Δ_{34} given by Eq.(25). Note that the minimum maintenance cost of each policy is numerically obtained by using the approaches presented in Appendix C when varying the decision variables's values.

Policy P1 vs Policy P2. Following Table 3, there are significant correlations between Δ_{12} and the two parameters k and c_i. Therefore, in this paragraph, we first investigate how much more benefit Policy P2 provides over Policy P1 with different values of k and c_i. The results are sketched in Figure 8.

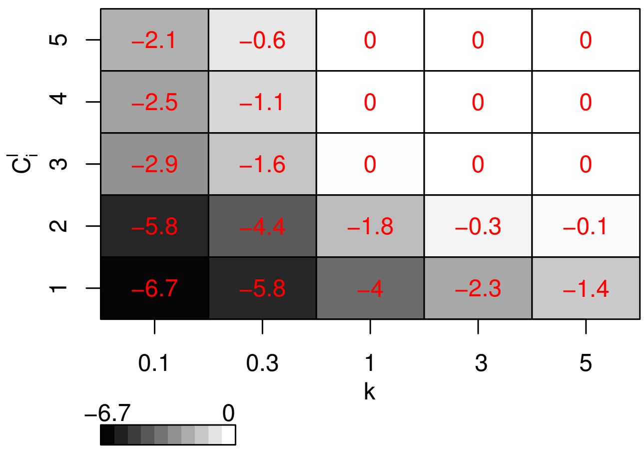


Figure 8: Performance of adjusted-quality inspections for fixed maintenance policy : Δ_{12} as a function of k (shape parameter of the inspection cost function) and c_i^l (cost of the low quality inspection) - With $c_d = 25$, $c_c = 100$, $c_p = 50$

Figure 8 represents the relative decrease ($\Delta_{12} < 0$) between the optimal costs of Policy P2 and Policy P1. In other words, it characterizes the benefit provided by adjusted-quality inspections when being integrated in the maintenance policy with fixed preventive replacement threshold. The shade of gray represents the value range of Δ_{12} : the darker, the greater the benefit of the adjusted-quality inspection option is. The horizontal axis represents the value of k - the parameter characterizing the shape of the inspection cost function (see Eq.(15)) while the vertical axis characterizes the cost of low quality inspection (c_i^l). Recall that if $0 < k < 1$, the inspection cost function is convex, whereas it is linear for $k = 1$, and concave for $k > 1$.

When k increases, the benefit of adjusted-quality inspections decreases. In other words, if the inspection cost decreases faster than the inspection quality, it is preferable to use adjusted-quality inspections for fixed maintenance policy.

Consider for example the first column of Figure 8, when $k = 0.1$, using adjusted-quality inspections helps to reduce from 2.1% to 6.7% of the cost incurred by fixed inspection policy. However, when the inspection cost decreases slower than the inspection quality, (i.e. $k = 3$ or 5), adjusted-quality inspection cannot provide more benefit when compared with the non-adjusted one ($\Delta_{12} = 0$), especially for high inspection cost. Indeed, the benefit of adjusted-quality inspections decreases when the inspection cost increases. Since we consider a constant ratio between the high and low inspection cost, $c_i^h/c_i^l = 3$, if $c_i^l = 5$, then $c_i^h = 15$, it is preferable to use only inspections with low quality, which explains why the benefit of adjusted-quality inspections decreases when k increases.

Proposed Policy P vs Policy P3. Figure 9 represents the relative decrease ($\Delta_{34} < 0$) in the maintenance cost incurred by the Proposed Policy P when compared to Policy P3. It characterizes the benefit provided by adjusted-quality inspections when being integrated in a dynamic maintenance policy (with replacement option at every inspection period). The smaller the negative value of Δ_{34} is, the greater the benefit that the Proposed Policy P can provide when used instead of Policy P3. The significations of color shades, horizontal and vertical axis are similar to the ones of Figure 8.

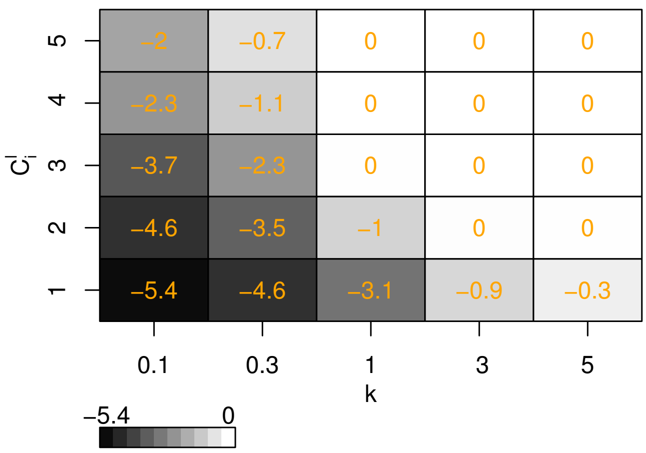


Figure 9: Performance of adjusted-quality inspections for dynamic replacement policy : Δ_{34} as a function of k (shape parameter of the inspection cost function) and c_i^l (cost of the low quality inspection) - With $c_d = 25$, $c_c = 100$, $c_p = 50$

In the most favorable configurations (not too expensive inspections and convex inspection cost function), the Proposed Policy P allows significant savings over Policy P3. The results show that when k increases, the benefit provided by the Proposed Policy significantly decreases. This benefit also decreases with c_i, showing that when the inspection cost is high, it is preferable to use low quality inspections. In this latter case, the adjusted-quality inspection policy tends to the fixed-quality inspection policy.

4.4.3. Dynamic inspection vs dynamic maintenance decision : performance comparison

Policy P2 consists in the combination of dynamic inspections and a fixed maintenance decision rule whereas Policy P3 combines fixed inspections and a dynamic maintenance decision rule. Corresponding to every combination of the input parameters (c_i^l, k, c_d, vc, etc.) we find the minimum maintenance cost of the maintenance policies P2, P3 and then calculate the percentage of their difference, e.g. Δ_{23} given by Eq.(25). Based on the evolution of Δ_{23} as a function of different system parameters, the aim of this section is to compare more in depth the behavior and the performance of these two policies P2 and P3 to gain a better understanding of their respective interests.

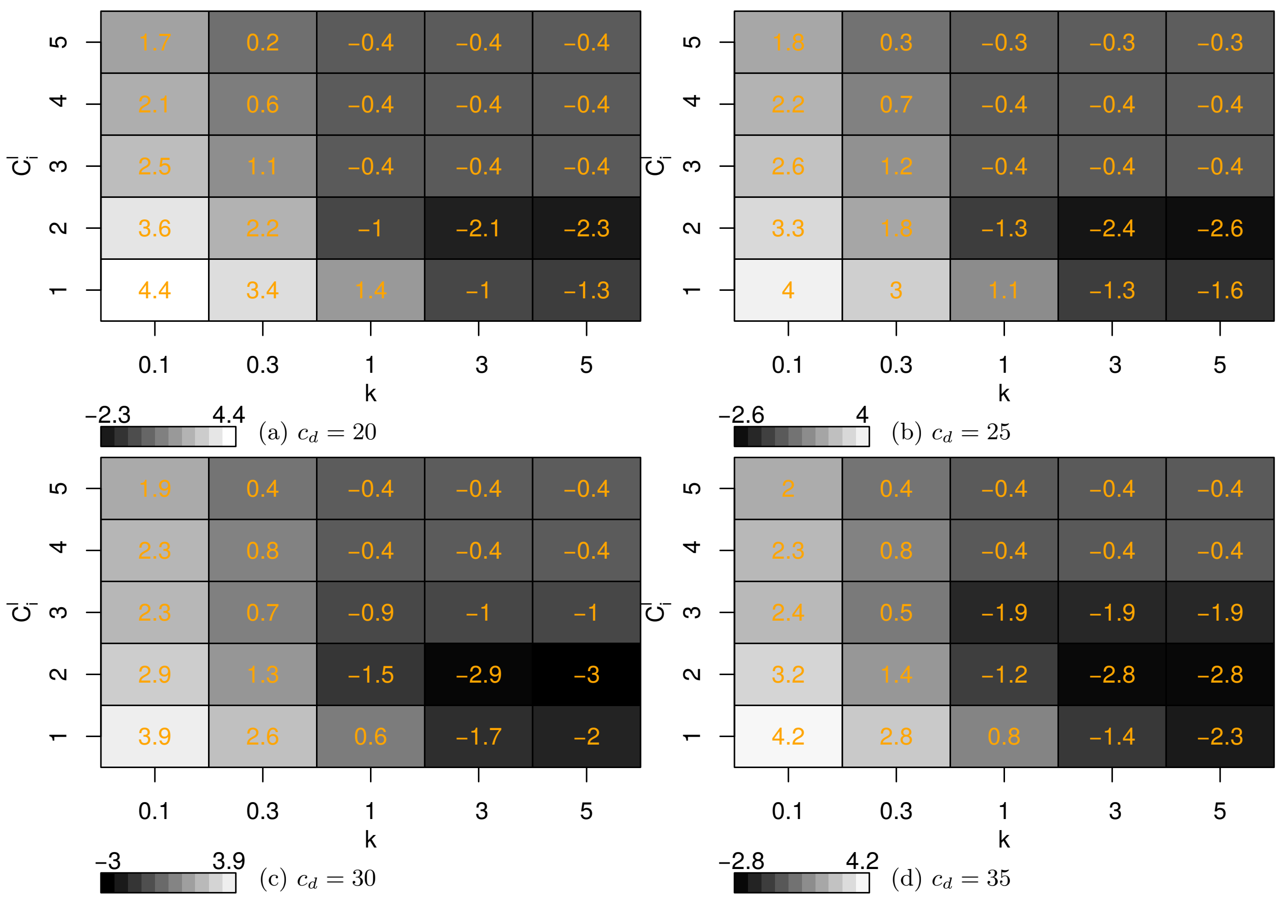


Figure 10: Comparison of Policy P2 and Policy P3: Δ_{23} as a function of k (shape parameter of the inspection cost function) and c_i^l (cost of the low quality inspection) - $c_c = 100$, $c_p = 50$ - For different values of c_d

Figure 10 presents the performance difference between Policy P3 and Policy P2 : Policy P2 is better than Policy P3 when $\Delta_{23} > 0$ and inversely. The values of Δ_{23} are then illustrated by shades of black: the darker, the smaller value of Δ_{23} is. It can be observed that:

- Δ_{23} is positive for all cases of k between $k = 0.1$ and 0.3, which indicates that the performance of the adjusted-quality policy is better than the one of the dynamic replacement policy when the inspection cost decreases faster than the inspection quality ($k < 1$).
- When $k = 1$, it is preferable to use Policy 2 with adjusted-quality inspections to reduce the maintenance cost for low inspection cost ($c_i^l = 1$). When the inspection cost increases, the performance of adjusted-quality inspection decreases, then the performance of Policy P3 is better than the one of Policy P2.
- When $k = 3$ or 5, the inspection cost increases faster than the inspection quality, and it is not possible to take advantage from an adjusted-quality inspection policy. In this case, it preferable to use Policy P3 with a dynamic replacement option to reduce the maintenance cost.

On the other hand, Figure 11 presents the evolution of Δ_{23} with the variance coefficient of the deterioration process vc and the shape coefficient of the inspection cost function k.

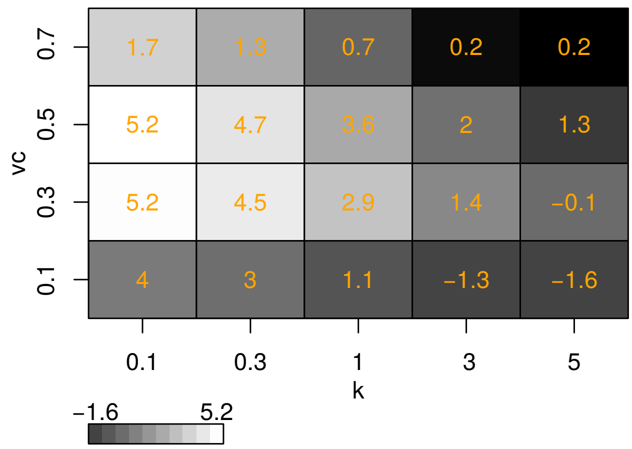


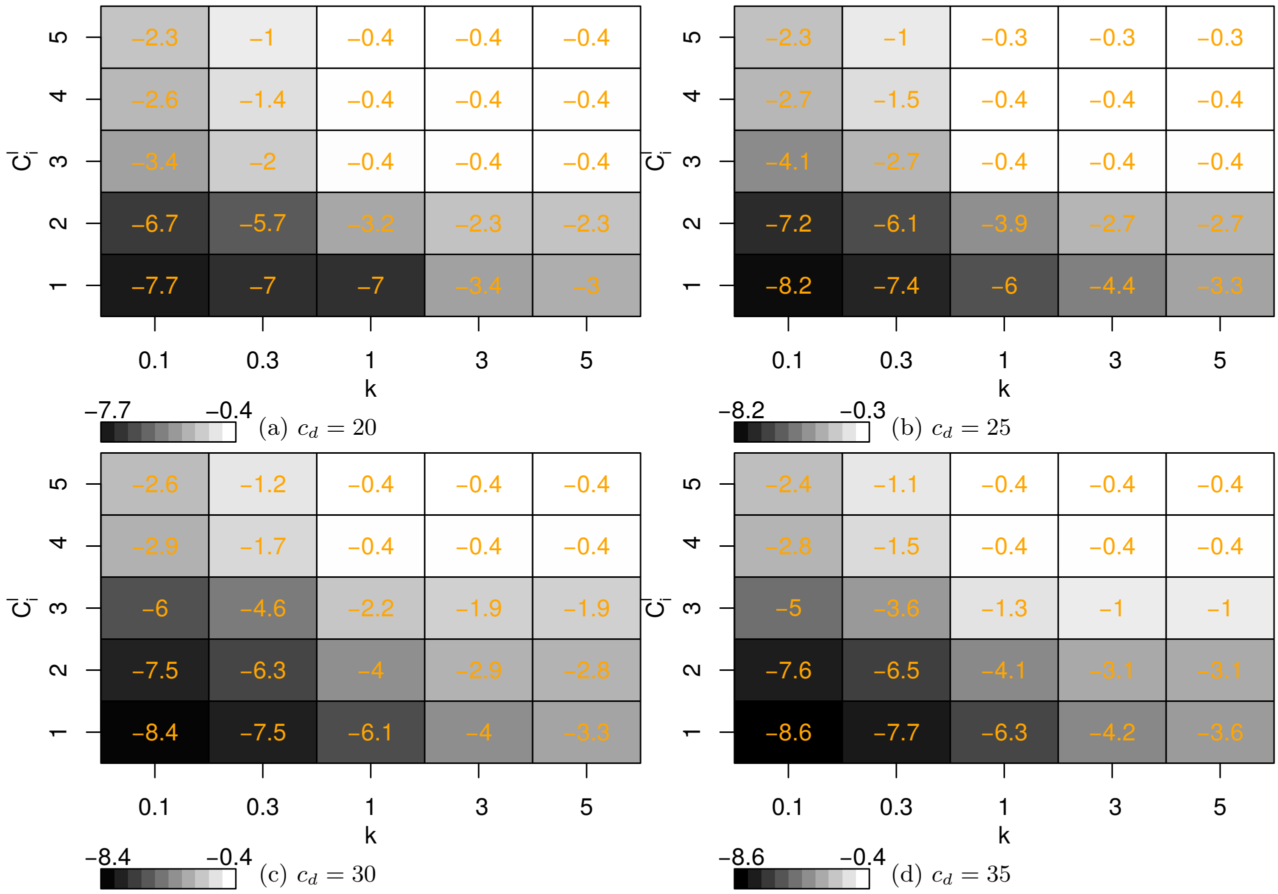
Figure 11: Comparison of Policy P2 and Policy P3 for different values of the variance coefficient vc and the shape parameter of the cost function k - With $c_d = 25$, $c_c = 100$, $c_p = 50$ and $c_i^l = 1$

It can be seen in Figure 11 that Δ_{23} is not monotone when vc increases, which explains why the correlation value of Δ_{23} and vc is low, see Table 3 . However, an interesting result can be recognized: when vc increases from 0.1 to 0.3 or 0.5, Δ_{23} increases as well. It is thus preferable to use Policy P2 with adjusted-quality-inspections than Policy P3 with fixed-quality-inspections when the variance coefficient vc of the degradation process increases. However, if vc is very high, e.g. $vc = 0.7$, the high quality inspections might not be necessary. Therefore, the performance of adjusting-quality-inspections decreases and then, the value of Δ_{23} also decreases when vc increases from 0.5 to 0.7.

4.4.4. Performance of dynamic inspection-maintenance policy

In this section, we investigate the performance gain brought by the dynamic inspection-maintenance policy (P) over the classic CBM. In detail, corresponding to every combination of the input parameters (c_d, k, c_i^l, etc.) we find the optimal values of the policies P and P1 and then calculate the percentage of their difference, e.g. Δ_{14} given by Eq.(25). The performance gain brought by the policy P over the policy P1 is represented in Figure 12. We find that in all cases, Δ_{14} is negative, the policy P is always better than Policy 1. In addition, the benefit of using P is more significant when the value of k is decreasing. In fact, when $c_i^l = 1$, $c_d = 35$ (sub figure 12d), using the dynamic maintenance inspection policy helps the manager to reduce 8.6% of the cost incurred by the fixed inspection maintenance policy if $k = 0.1$ while it is only 3.6% if $k = 5$.

On the other hand, we recognize the performance gain brought by the policy P is increasing in c_d. In fact, in order to avoid an important down-time cost when c_d increases, it prefers high quality inspections



(a) $c_d = 20$ (b) $c_d = 25$ (c) $c_d = 30$ (d) $c_d = 35$

Figure 12: Comparison of Policy P and Policy P1: Δ_{14} as a function of k (shape parameter of the inspection cost function) and c_i^l (cost of the low quality inspection) - $c_c = 100$, $c_p = 50$ - For different values of c_d.

with dynamic maintenance option than the classic CBM. And therefore, a dynamic inspection-maintenance policy that allows using acceptable quality inspections with low cost will provide more benefit in these cases.

5. Conclusion

In this paper, we propose a dynamic condition-based maintenance and monitoring policy using POMDP model for a system subject to a continuous degradation process and imperfect inspection representative by observation noises. We firstly develop some discretization formulations that allow applying a POMDP model for a continuous degradation process. More importantly, this allows removing the barrier of the assumption that the conditional probability of the discrete observation given the system state is known. The impacts of imperfect inspection are described by Gaussian distribution. Different inspections quality levels are considered and investigated regarding not only their impacts on observation noises and their related cost. A cost model is then proposed to evaluate the performance of the proposed maintenance and monitoring policy. The performance of the Proposed Policy is highlighted through numerical examples. It is compared to the performance of different inspection-maintenance policies: 1) a classical condition-based inspection and replacement policy, 2) a fixed replacement policy with adjusted-quality inspections, 3) a dynamic inspection policy with non-adjusted-quality inspections. Thanks to the flexibility introduced by the adaptive inspection quality, the proposed dynamic policy gives better results than the more classical static one. When the inspection cost decreases faster than inspection quality, the benefit of adjusted-quality inspections for maintenance policy is important. Otherwise, the benefit is non significant when the inspection cost decreases less rapidly than the inspection quality.

This work focuses on the interest of adjustment of inspection quality in maintenance optimization. Therefore, we have only considered perfect maintenance actions, and do not investigate imperfect maintenances. It can be seen as a limitation of our model. In further work, we plan to develop a more flexible model that allows us to consider multiple options for maintenance and also to schedule dynamically the next inspection time. In addition, working all along the model development with the continuous deterioration should lead to better maintenance performance in theory, but may be more difficult to implement in practice. Hence, the impact of the discretization of the continuous process on the performance optimal policies and the trade-off resulting from this discretization could be investigated in detail.

Acknowledgements

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Appendix

Appendix A. Formulation of $P\left(Z_{T_{n+1}} = m \mid Z_{T_n} = k\right)$

We assume that $\{X_t\}_{t\geq 0}$ evolves monotonically according to a homogeneous gamma process with shape parameter $\alpha > 0$ and scale parameter $\beta > 0$. This means the system degradation increment between two instants s and t, $X_t - X_s$, is gamma distributed with probability density function (pdf):

$$f_{\alpha\cdot(t-s),\beta}\left(x\right) = \frac{1}{\Gamma(\alpha)} \cdot (\beta)^{\alpha} \cdot x^{\alpha-1} \cdot e^{-\beta\cdot x} \cdot \mathcal{I}_{\{x\geq 0\}},$$

where:

- $\Gamma(\alpha) = \int\limits_0^{+\infty} u^{\alpha-1} \cdot e^{-u} du$ denotes a complete gamma function;
- $\mathcal{I}_{\{x\geq 0\}}$ is an indicator function. $\mathcal{I}_{\{x\geq 0\}} = 1$ if $x \geq 0$, $\mathcal{I}_{\{x\geq 0\}} = 0$ and otherwise.

In that way, $P\left(Z_{T_{n+1}}=m \mid Z_{T_n}=k\right)$ can be calculated as follows:

- If $1 \le k = m \le N$,

$$\begin{aligned}
&P\left(Z_{T_{n+1}}=k \mid Z_{T_n}=k\right)=P\left((k-1)\,l \le X_{T_{n+1}}<kl \mid (k-1)\,l \le X_{T_n}<kl\right)\\
&=\frac{P\left((k-1)\,l \le X_{T_{n+1}}<kl,(k-1)\,l \le X_{T_n}<kl\right)}{P\left((k-1)\,l \le X_{T_n}<kl\right)}=\frac{P\left((k-1)\,l \le X_{T_n}<X_{T_{n+1}}<kl\right)}{F_{\alpha\cdot T_n,\beta}\left(kl\right)-F_{\alpha\cdot T_n,\beta}\left((k-1)\,l\right)}\\
&=\frac{\int_{(k-1)l}^{kl} P\left(X_{T_{n+1}}<kl \mid X_{T_n}=x\right) f_{\alpha\cdot T_n,\beta}\left(x\right)dx}{F_{\alpha\cdot T_n,\beta}\left(kl\right)-F_{\alpha\cdot T_n,\beta}\left((k-1)\,l\right)}=\frac{\int_{(k-1)l}^{kl} P\left(X_{T_{n+1}}-X_{T_n}<kl-x\right) f_{\alpha\cdot T_n,\beta}\left(x\right)dx}{F_{\alpha\cdot T_n,\beta}\left(kl\right)-F_{\alpha\cdot T_n,\beta}\left((k-1)\,l\right)}\\
&=\frac{\int_{(k-1)l}^{kl} F_{\alpha\cdot(T_{n+1}-T_n),\beta}\left(kl-x\right) f_{\alpha\cdot T_n,\beta}\left(x\right)dx}{F_{\alpha\cdot T_n,\beta}\left(kl\right)-F_{\alpha\cdot T_n,\beta}\left((k-1)\,l\right)},
\end{aligned}$$

- If $1 \le k < m \le N$,

$$\begin{aligned}
&P\left(Z_{T_{n+1}}=m \mid Z_{T_n}=k\right)=P\left((m-1)\,l \le X_{T_{n+1}}<ml \mid (k-1)\,l \le X_{T_n}<kl\right)\\
&=\frac{P\left((m-1)\,l \le X_{T_{n+1}}<ml,(k-1)\,l \le X_{T_n}<kl\right)}{P\left((k-1)\,l \le X_{T_n}<kl\right)}=\frac{P\left((k-1)\,l \le X_{T_n}<kl \le (m-1)\,l \le X_{T_{n+1}}<ml\right)}{F_{\alpha\cdot T_n,\beta}\left(kl\right)-F_{\alpha\cdot T_n,\beta}\left((k-1)\,l\right)}\\
&=\frac{\int_{(k-1)l}^{kl} P\left((m-1)\,l \le X_{T_{n+1}}<ml \mid X_{T_n}=x\right) f_{\alpha\cdot T_n,\beta}\left(x\right)dx}{F_{\alpha\cdot T_n,\beta}\left(kl\right)-F_{\alpha\cdot T_n,\beta}\left((k-1)\,l\right)}\\
&=\frac{\int_{(k-1)l}^{kl} P\left((m-1)\,l-x \le X_{T_{n+1}}-X_{T_n}<ml-x\right) f_{\alpha\cdot T_n,\beta}\left(x\right)dx}{F_{\alpha\cdot T_n,\beta}\left(kl\right)-F_{\alpha\cdot T_n,\beta}\left((k-1)\,l\right)}\\
&=\frac{\int_{(k-1)l}^{kl} \left(F_{\alpha\cdot(T_{n+1}-T_n),\beta}\left(ml-x\right)-F_{\alpha\cdot(T_{n+1}-T_n),\beta}\left((m-1)\,l-x\right)\right) f_{\alpha\cdot T_n,\beta}\left(x\right)dx}{F_{\alpha\cdot T_n,\beta}\left(kl\right)-F_{\alpha\cdot T_n,\beta}\left((k-1)\,l\right)},
\end{aligned}$$

- If $1 \le k \le N$, $m = N+1$,

$$\begin{aligned}
&P\left(Z_{T_{n+1}}=N+1 \mid Z_{T_n}=k\right)=P\left(L \le X_{T_{n+1}} \mid (k-1)\,l \le X_{T_n}<kl\right)\\
&=\frac{P\left(L \le X_{T_{n+1}},(k-1)\,l \le X_{T_n}<kl\right)}{P\left((k-1)\,l \le X_{T_n}<kl\right)}=\frac{P\left((k-1)\,l \le X_{T_n}<kl \le L \le X_{T_{n+1}}\right)}{F_{\alpha\cdot T_n,\beta}\left(kl\right)-F_{\alpha\cdot T_n,\beta}\left((k-1)\,l\right)}\\
&=\frac{\int_{(k-1)l}^{kl} P\left(L \le X_{T_{n+1}} \mid X_{T_n}=x\right) f_{\alpha\cdot T_n,\beta}\left(x\right)dx}{F_{\alpha\cdot T_n,\beta}\left(kl\right)-F_{\alpha\cdot T_n,\beta}\left((k-1)\,l\right)}=\frac{\int_{(k-1)l}^{kl} P\left(X_{T_{n+1}}-X_{T_n}\ge L-x\right) f_{\alpha\cdot T_n,\beta}\left(x\right)dx}{F_{\alpha\cdot T_n,\beta}\left(kl\right)-F_{\alpha\cdot T_n,\beta}\left((k-1)\,l\right)}\\
&=\frac{\int_{(k-1)l}^{kl} \left(1-F_{\alpha\cdot(T_{n+1}-T_n),\beta}\left(L-x\right)\right) f_{\alpha\cdot T_n,\beta}\left(x\right)dx}{F_{\alpha\cdot T_n,\beta}\left(kl\right)-F_{\alpha\cdot T_n,\beta}\left((k-1)\,l\right)},
\end{aligned}$$

- If $k = N+1$, $m = N+1$,

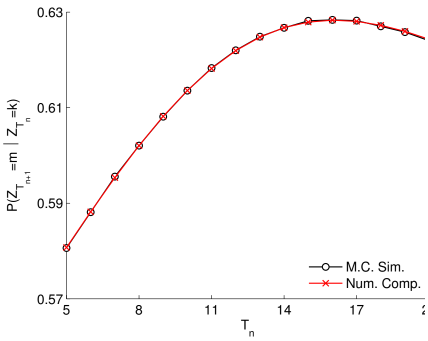
$$P\left(Z_{T_{n+1}}=N+1 \mid Z_{T_n}=N+1\right)=P\left(L \le X_{T_{n+1}} \mid L \le X_{T_n}\right)=1.$$

Verification of $P\left(Z_{T_{n+1}}=m \mid Z_{T_n}=k\right)$

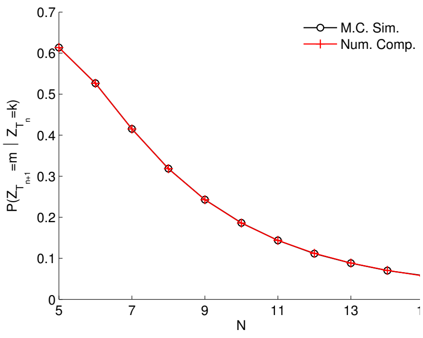
To verify the exactness of the expressions of $P\left(Z_{T_{n+1}}=m \mid Z_{T_n}=k\right)$, we consider a system whose the deterioration/failure process is defined by the set of parameters $\alpha = 1$, $\beta = 1$, $L = 15$, and the noisy observation is characterized by $\sigma_q = 1$. Then, we compute numerically $P\left(Z_{T_{n+1}} = m \mid Z_{T_n} = k\right)$ and compare with the results returned by the Monte Carlo simulation under three different configurations:

1. *varied n-th inspection time* T_n: T_n varies from 5 to 20 with step 1, $N = 5$, and $T = 4$,
2. *varied discretized number of states* N: $T_n = 10$, N varies from 5 to 15 with step 1, and $T = 4$,
3. *varied inspection period* T: $T_n = 10$, $N = 5$, and T varies from 1 to 9 with step 1.

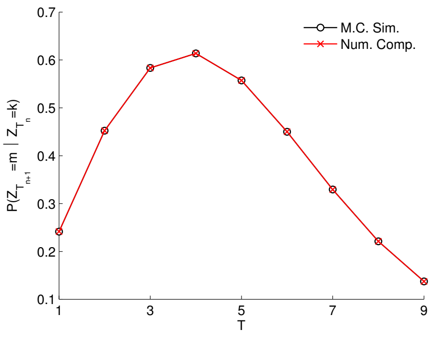
The states m and k are chosen as 4 and 3 respectively, which corresponds the second case of $P\left(Z_{T_{n+1}} = m \mid Z_{T_n} = k\right)$ (i.e., $1 \leq k < m \leq N$). The other cases can be verified in the same manner, and are not represented here. The integral in the expression of $P\left(Z_{T_{n+1}} = m \mid Z_{T_n} = k\right)$ is numerically evaluated by `integrate` R function, and the number of histories for Monte Carlo Simulation is $N_h = 5 \times 10^7$. The results of the three above configurations are represented in Figure ??, ?? and ?? respectively.



(a) Varied T_n



(b) Varied N



(c) Varied T

Figure 13: Numerical verification of $P\left(Z_{T_{n+1}} = m \mid Z_{T_n} = k\right)$

The identical curves given by the numerical computation (cross red lines) and by the Monte Carlo simulation (circle black lines) justify the correctness of the expressions of $P\left(Z_{T_{n+1}} = m \mid Z_{T_n} = k\right)$.

Appendix B. Formulation of $P\left(O_n = h \mid Z_{T_n} = k\right)$

In the case where $\{X_t\}_{t \geq 0}$ evolves monotonically according to a homogeneous gamma process with shape parameter $\alpha > 0$ and scale parameter $\beta > 0$, the system degradation increment between two instants s and t, $X_t - X_s$, is gamma distributed with probability density function (pdf):

$$f_{\alpha \cdot (t-s), \beta}\left(x\right) = \frac{1}{\Gamma(\alpha)} \cdot \left(\beta\right)^{\alpha} \cdot x^{\alpha - 1} \cdot e^{-\beta \cdot x} \cdot \mathcal{I}_{\{x \geq 0\}},$$

where:

- $\Gamma(\alpha) = \int\limits_{0}^{+\infty} u^{\alpha-1} \cdot e^{-u} du$ denotes a complete gamma function;

- $\mathcal{I}_{\{x\geq 0\}}$ is an indicator function. $\mathcal{I}_{\{x\geq 0\}}=1$ if $x\geq 0$, $\mathcal{I}_{\{x\geq 0\}}=0$ and otherwise.

In this case, $P\left(O_n=h\mid Z_{T_n}=k\right)$ can be written as follows

- If $h\in[2,N]$ and $k\in[1,N]$,

$$\begin{aligned}
P\left(O_n=h\mid Z_{T_n}=k\right) &= P\left((h-1)\,l\leq Y_{T_n}<hl\mid (k-1)\,l\leq X_{T_n}<kl\right)\\
&= \frac{P\left((h-1)\,l\leq Y_{T_n}<hl,(k-1)\,l\leq X_{T_n}<kl\right)}{P\left((k-1)\,l\leq X_{T_n}<kl\right)}\\
&= \frac{P\left((h-1)\,l\leq X_{T_n}+E_q<hl,(k-1)\,l\leq X_{T_n}<kl\right)}{F_{\alpha\cdot T_n,\beta}\left(kl\right)-F_{\alpha\cdot T_n,\beta}\left((k-1)\,l\right)}\\
&= \frac{\int_{(k-1)l}^{kl}P\left((h-1)\,l\leq X_{T_n}+E_q<hl\mid X_{T_n}=x\right)f_{\alpha\cdot T_n,\beta}\left(x\right)dx}{F_{\alpha\cdot T_n,\beta}\left(kl\right)-F_{\alpha\cdot T_n,\beta}\left((k-1)\,l\right)}\\
&= \frac{\int_{(k-1)l}^{kl}P\left((h-1)\,l-x\leq E_q<hl-x\right)f_{\alpha\cdot T_n,\beta}\left(x\right)dx}{F_{\alpha\cdot T_n,\beta}\left(kl\right)-F_{\alpha\cdot T_n,\beta}\left((k-1)\,l\right)}\\
&= \frac{\int_{(k-1)l}^{kl}\left(G_{\sigma_q}\left(hl-x\right)-G_{\sigma_q}\left((h-1)\,l-x\right)\right)f_{\alpha\cdot T_n,\beta}\left(x\right)dx}{F_{\alpha\cdot T_n,\beta}\left(kl\right)-F_{\alpha\cdot T_n,\beta}\left((k-1)\,l\right)},
\end{aligned}$$

- If $h=N+1$ and $k\in[1,N]$,

$$\begin{aligned}
P\left(O_n=N+1\mid Z_{T_n}=k\right) &= P\left(L\leq Y_{T_n}\mid (k-1)\,l\leq X_{T_n}<kl\right)\\
&= \frac{P\left(L\leq Y_{T_n},(k-1)\,l\leq X_{T_n}<kl\right)}{P\left((k-1)\,l\leq X_{T_n}<kl\right)}\\
&= \frac{P\left(L\leq X_{T_n}+E_q,(k-1)\,l\leq X_{T_n}<kl\right)}{F_{\alpha\cdot T_n,\beta}\left(kl\right)-F_{\alpha\cdot T_n,\beta}\left((k-1)\,l\right)}\\
&= \frac{\int_{(k-1)l}^{kl}P\left(L\leq X_{T_n}+E_q\mid X_{T_n}=x\right)f_{\alpha\cdot T_n,\beta}\left(x\right)dx}{F_{\alpha\cdot T_n,\beta}\left(kl\right)-F_{\alpha\cdot T_n,\beta}\left((k-1)\,l\right)}\\
&= \frac{\int_{(k-1)l}^{kl}P\left(L-x\leq E_q\right)f_{\alpha\cdot T_n,\beta}\left(x\right)dx}{F_{\alpha\cdot T_n,\beta}\left(kl\right)-F_{\alpha\cdot T_n,\beta}\left((k-1)\,l\right)}\\
&= \frac{\int_{(k-1)l}^{kl}\left(1-G_{\sigma_q}\left(L-x\right)\right)f_{\alpha\cdot T_n,\beta}\left(x\right)dx}{F_{\alpha\cdot T_n,\beta}\left(kl\right)-F_{\alpha\cdot T_n,\beta}\left((k-1)\,l\right)},
\end{aligned}$$

- If $h\in[2,N]$ and $k=N+1$,

$$\begin{aligned}
&P\left(O_n=h\mid Z_{T_n}=N+1\right)=P\left((h-1)\,l\leq Y_{T_n}<hl\mid L\leq X_{T_n}\right)=\frac{P\left((h-1)\,l\leq Y_{T_n}<hl,L\leq X_{T_n}\right)}{P\left(L\leq X_{T_n}\right)}\\
&=\frac{P\left((h-1)\,l\leq X_{T_n}+E_q<hl,L\leq X_{T_n}\right)}{1-F_{\alpha\cdot T_n,\beta}\left(L\right)}=\frac{\int_{L}^{\infty}P\left((h-1)\,l\leq X_{T_n}+E_q<hl\mid X_{T_n}=x\right)f_{\alpha\cdot T_n,\beta}\left(x\right)dx}{1-F_{\alpha\cdot T_n,\beta}\left(L\right)}\\
&=\frac{\int_{L}^{\infty}P\left((h-1)\,l-x\leq E_q<hl-x\right)f_{\alpha\cdot T_n,\beta}\left(x\right)dx}{1-F_{\alpha\cdot T_n,\beta}\left(L\right)}=\frac{\int_{L}^{\infty}\left(G_{\sigma_q}\left(hl-x\right)-G_{\sigma_q}\left((h-1)\,l-x\right)\right)f_{\alpha\cdot T_n,\beta}\left(x\right)dx}{1-F_{\alpha\cdot T_n,\beta}\left(L\right)},
\end{aligned}$$

- If $h = N+1$ and $k = N+1$,

$$
\begin{aligned}
P\left(O_n = N+1 \mid Z_{T_n} = N+1\right) &= P\left(L \leq Y_{T_n} \mid L \leq X_{T_n}\right) = \frac{P\left(L \leq Y_{T_n}, L \leq X_{T_n}\right)}{P\left(L \leq X_{T_n}\right)} \\
&= \frac{P\left(L \leq X_{T_n} + E_q, L \leq X_{T_n}\right)}{P\left(L \leq X_{T_n}\right)} = \frac{\int_L^\infty P\left(L \leq X_{T_n} + E_q \mid X_{T_n} = x\right) f_{\alpha \cdot T_n, \beta}(x)\, dx}{1 - F_{\alpha \cdot T_n, \beta}(L)} \\
&= \frac{\int_L^\infty P\left(L - x \leq E_q\right) f_{\alpha \cdot T_n, \beta}(x)\, dx}{1 - F_{\alpha \cdot T_n, \beta}(L)} = \frac{\int_L^\infty \left(1 - G_{\sigma_q}(L - x)\right) f_{\alpha \cdot T_n, \beta}(x)\, dx}{1 - F_{\alpha \cdot T_n, \beta}(L)},
\end{aligned}
$$

- If $h = 1$ and $k \in [1, N+1]$,

$$
P\left(O_n = 1 \mid Z_{T_n} = k\right) = 1 - \sum_{h=2}^{N+1} P\left(O_n = h \mid Z_{T_n} = k\right).
$$

Verification of $P\left(O_n = h \mid Z_{T_n} = k\right)$

To verify the expression of $P\left(O_n = h \mid Z_{T_n} = k\right)$, a similar approach as above is used. Indeed, we always consider a system with parameters $\alpha = 1$, $\beta = 1$, $L = 15$, choose the inspection period $T = 4$, and then study three different configurations:

1. *varied n-th inspection time* T_n: T_n varies from 5 to 20 with step 1, $N = 5$, and $\sigma_q = 1$,
2. *varied discretized number of states* N: $T_n = 10$, N varies from 5 to 15 with step 1, and $\sigma_q = 1$,
3. *varied standard deviation of Gaussian observation errors* σ_q: $T_n = 10$, $N = 5$, and σ_q varies from 0 to 6 with step 0.25.

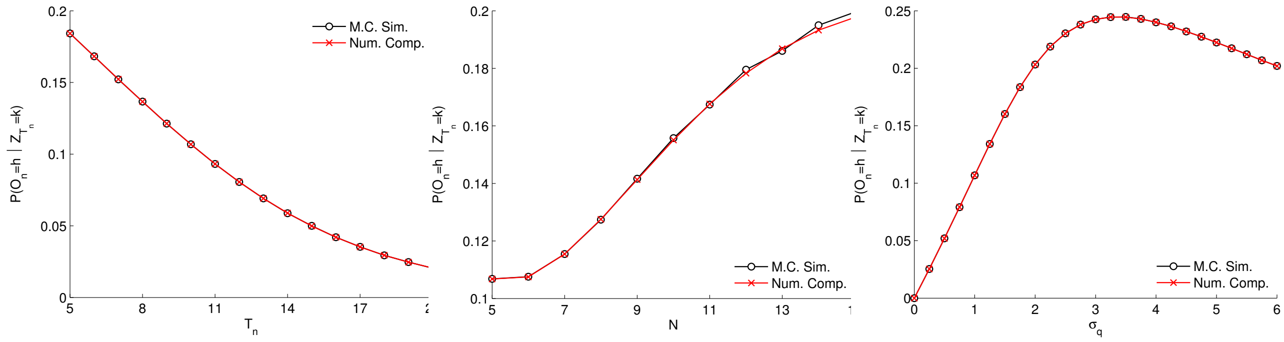
The states h and k are chosen as 2 and 3 respectively, which corresponds the first case of $P\left(O_n = h \mid Z_{T_n} = k\right)$ (i.e., $h \in [2, N]$ and $k \in [1, N]$). The other cases can be verified in the same manner, and are not shown here for a concise representation. Figures 14a, 14b and 14c show the results of the three above configurations.

Once again, the identical curves given by both the numerical computation and Monte Carlo simulation justify the exactness of the expressions of $P\left(O_n = h \mid Z_{T_n} = k\right)$.

Appendix C. Formulation of policies P1, P2 and P3

Formulation of Policy 3:

For policy P3, we firstly decide the inspection period length T throughout the planning horizon. Then, associated to T, at every inspection period, preventive replacement (R) or do nothing decision (DN) is effected based on the updated belief function b_{T_n} after non-adjusted-quality-inspections. For PO_3^l-policy, low quality inspections will be effected throughout the planning horizon. For PO_3^h-policy, high quality



(a) Varied T_n (b) Varied N (c) Varied σ_q

Figure 14: Numerical verification of $P\left(O_n = h \mid Z_{T_n} = k\right)$

inspections will be effected throughout the planning horizon. Then the formulation of Policy P3 is similar to the one of the proposed policy but Eq.(19) is re-written as follows:

$$V_{[T_n,T_{end}]}(T, b_{T_n}) = \mathbb{P}_f(T_n)C_C(\cdot) + \bar{\mathbb{P}}_f(T_n) \cdot \left(c_i^q + \sum_{h\in\mathbf{S}_O} P_q(O_n = h) \cdot \min\left[C_R(\cdot), \quad C_{DN}(\cdot)\right] \right) \tag{26}$$

where $c_i^q = c_i^h$, $\sigma_q = \sigma_h$ for PO_3^h-policy and $c_i^q = c_i^l$, $\sigma_q = \sigma_l$ for PO_3^l-policy.

The optimal value of Policy P3 with high (or low) quality inspections is given by the minimal value of the expected accumulated cost $V_0^T(\cdot)$ associated to different values of inspection period length T.

$$\begin{aligned} V_0^{P_3^h} &= \min_T \left(V_{[0,T_{end}]}(T, b_0)\right); \qquad \textit{in which, } c_i^q = c_i^h, \sigma_q = \sigma_h \\ V_0^{P_3^l} &= \min_T \left(V_{[0,T_{end}]}(T, b_0)\right); \qquad \textit{in which, } c_i^q = c_i^l, \sigma_q = \sigma_l \end{aligned} \tag{27}$$

Formulation of Policy P2:

For Policy P2, we firstly decide the inspection period length T and the preventive replacement threshold M that is used throughout the planning horizon. Then, preventive replacement is decided directly based on the observation state O_n,

- If $O_n \geq M$ the system is preventively replaced by a new one with cost c_p.
- If system is failed, it is correctively replaced by a new one with cost c_c ($c_c > c_p$).

The formulation of Policy P2 is similar to the one of the proposed policy but Eq.(19) is re-written as follows:

$$V_{[T_n,T_{end}]}(T, b_{T_n}) = \mathbb{P}_f(T_n)C_C(\cdot) + \bar{\mathbb{P}}_f(T_n) \cdot \min_q \left(c_i^q + \sum_{h=M}^{z} P_q(O_n = h)C_R(\cdot) + \sum_{h=1}^{M-1} P_q(O_n = h)C_{DN}(\cdot) \right) \tag{28}$$

It is evaluated by the sum of the expected accumulated cost associated to system failure $C_C(\cdot)$ and the one associated to system non-failure. At the beginning of the observation period T_n, if system still works, it is necessary to decide the quality level for inspection and then, the maintenance action (preventive replace or not) based on the observation (O_n). The quality index q is chosen such as the expected cost accumulated is minimal. It is evaluated by the inspection cost c_i^q, the expected accumulated cost $C_R(\cdot)$ corresponding to the preventive replacement decision if $O_n \geq M$ and the expected accumulated cost $C_{DN}(\cdot)$ corresponding to the operation decision without replacement when $O_n < M$.

The optimal value of Policy P2, $V_0^{P_2}$ is given by the minimal value of the expected accumulated cost $V_{[0,T_{end}]}(T, b_0)$ associated to different values of decision variable couple (T, M).

Formulation of Policy 1:

Similar to Policy P2, we firstly decide the inspection period length T and the preventive replacement threshold M that is used throughout the planning horizon. Then, preventive replacement is decided directly based on the observation state O_n: the system is preventively replaced when observation is superior than preventive replacement threshold $O_n \geq M$. For the failure case, it is correctively replaced. The formulation of Policy 1 is similar to the one of the proposed but Eq.(19) is re-written as follows:

$$V_0^{P_1}(T_n, M) = \mathbb{P}_f(T_n)C_C(\cdot) + \bar{\mathbb{P}}_f(T_n) \cdot \left(c_i^q + \sum_{h=M}^{z} P_q(O_n = h)C_R(\cdot) + \sum_{h=1}^{M-1} P_q(O_n = h)C_{DN}(\cdot) \right) \quad (29)$$

It is similar to Eq.(28) but without inspection quality adjustments. For P_1^l-policy, low quality inspections will be effected throughout the planning horizon ($\sigma_q = \sigma_l$, $c_i^q = c_i^l$). For P_1^h-policy, high quality inspections will be effected throughout the planning horizon ($\sigma_q = \sigma_h$, $c_i^q = c_i^h$).

The optimal value of Policy P1 with high quality inspections $V_0^{P_1^h}$ or the one with low quality inspections $V_0^{P_1^l}$ is given by the minimal value of the expected accumulated cost associated to different values of (T, M).

Approach for the determination of the optimal policies:

The main contribution of this paper is not the development of a new optimization algorithm and we use a classical grid based search to find the optimal strategies. In principle, the cost model formulations of the three policies P1, P2 and P3 (presented in Appendix C) are used to evaluate the maintenance policy cost corresponding to every combination of the decision variables. Then, the combination that provides the minimal value is retained. This process is easy to implement, even if it may be computationally burdensome.

- **Policy 1:** The cost $V_0^{P_1}(T, M, q)$ over the planning horizon $[0, T_{end}]$ where $T_{end} = 30$ is evaluated for every value of the inspection period length T, $T \in [1, 2, 3, ..., 15]$, for every value of the preventive replacement threshold M, $M \in [1, 2, 3, ..., 5]$ and for every value of the inspection quality $q \in [1, 6]$. The optimal combination, which provides the minimal value, is $(q^*, T^*, M^*) = (6, 1, 5)$.

- **Policy 2:** The cost $V_0^{P_2}(T, M)$ over the planning horizon $[0, T_{end}]$ where $T_{end} = 30$ is evaluated for every value of the inspection period length T, $T \in [1, 2, 3, ..., 15]$, and for every value of the preventive replacement threshold M, $M \in [1, 2, 3, ..., 5]$. Note that $V_0^{P_2}$ is the optimal value of the adjusted-quality-inspection process, in which the inspection quality is decided at the beginning of every decision period T_n. This optimal decision, q^* at T_n, is the one that minimizes the accumulated cost from T_n to T_{end}, given by Eq. (28). Finally, the combination of the decision variables (T, M) and the adjusted-quality-inspection process that provides the minimal value will be chosen.

- **Policy 3:** The cost $V_0^{P_3}(T, q)$ over the planning horizon $[0, T_{end}]$ where $T_{end} = 30$ is evaluated for every value of the inspection period length T, $T \in [1, 2, 3, ..., 15]$, and for every value of the inspection quality $q \in [1, 6]$. Note that $V_0^{P_3}(T, q)$ (with low or high quality inspection) is the optimal value of the maintenance decision process, in which the option (replacement or not) is decided at the beginning of every decision period T_n. This optimal decision is the one that minimizes the accumulated cost from T_n to T_{end}, given by Eq. (26). Finally, the combination of the decision variables (T, q) and the maintenance decision process that provides the minimal value will be chosen.

The optimal decision process, which minimizes the $V_0^{P_2}(T, M)$ or $V_0^{P_3}(T, q)$, is solved as a POMDP, similar to the policy P. In this paper, a fixed grid approximation approach is used to solve the POMDP. This approach include the following phases:

- **Phase 1 - Generate points of the belief grid**.

  We use the Monte Carlo Simulation approach to generate the single-step forward trajectories of the belief function b from the initial period to the 10-th period with random observations and actions (the approach SSRA presented in [26]). The acquired belief values are sorted following the likelihood ratio order, i.e. $b2(x) \succ_{LR} b1(x) \Rightarrow \sum_x b2(x)\phi(x) \geq \sum_x b1(x)\phi(x)$ for non-decreasing functions $\phi(x)$. To assure the accuracy of solutions, an enough large space of approximated belief values, B, which includes 3003 values, is investigated. As the number of belief points is large, the computational cost is high.

- **Phase 2 - Solve the problem using backward induction algorithm.**

  – We use the terminal condition of the planning horizon $V_{T_{end}} = 0$ to evaluate the policy value, $V^P_{(T_{end}-1)}$ for every point b of belief function space B and for every action of the action set A by Eqs. (19 - 22).

  – The optimal action at the period $(T_{end} - 1)$ for the belief b is the one that minimizes $V^P_{(T_{end}-1)}(b)$.

  – Similarly, the accumulated policy value, $V^P_{T_n}(b, a)$, from the n-th period to T_{end} is evaluated based on $V^P_{T_{n+1}}(b_{n+1})$, where b_{n+1} is the belief at the next period, i.e. $b_{n+1} = \tau(b, a, O_n)$ the transition of belief function b after performing action a and obtaining observation O_n.

  Note that the $V^P_{T_{n+1}}(b_{n+1})$ is approximated by $V^P_{T_{n+1}}(b)$ where b is a fixed point in the belief grid B, and also the nearest neighbor of b_{n+1} based on Euclidean distance. If in the grid, there are k points, which are close together following the likelihood ratio (LR) order, having the same minimal distance to b_{n+1}, the value $V^P_{T_{n+1}}(b_{n+1})$ is approximated by the mean value of $V^P_{T_{n+1}}$ of these points.

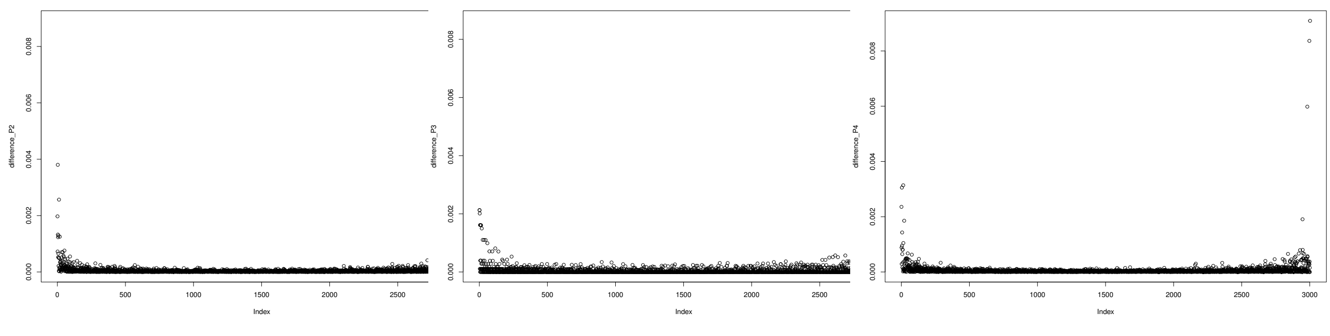
- The optimal action at the period T_n for the belief b is the one that minimizes $V^P_{T_n}(b)$.
- Finally, the optimal strategy, which allows minimizing the policy value $V^P_{T_0}$ is the optimal action process from the initial period T_0 to the end of the planning horizon T_{end}.

For an illustration of the interpolation between the grid points, let's assume that the belief at (n+1)-th decision period is $b_{n+1} = [0.69, 0.29, 0.01, 0.01, 0]$; and that its nearest neighbor in the grid is $b = [0.7, 0.3, 0, 0, 0]$. Then, the policy value at b_{n+1} can be approximated by the one at b.

Consider another example, let's assume that the belief at (n+1)-th decision period is $b_{n+1} = [0.9, 0.05, 0.05, 0, 0]$; and that its nearest neighbors in the grid are $b1 = [0.9, 0.1, 0, 0, 0]$ and $b2 = [0.9, 0, 0.1, 0, 0]$, $b2(x) \succ_{LR} b1(x)$, then the policy value at b_{n+1} can be approximated by the mean value of $V^P_{T_{n+1}}$ at two points $b1$ and $b2$.

Using the interpolation method presented in the above paragraphs, the accuracy of solutions depends on the differences of the policy values of nearest neighbor points in the belief grid. If these differences are insignificant, the solution error can be negligible. Let bi and bj be two consecutive points following the LR order in the grid ($bi(x) \succ_{LR} bj(x)$), the relative differences of their policy values are evaluated by:

$$\Delta_{V^P_{T_n}} = \frac{V^P_{T_n}(bj) - V^P_{T_n}(bi)}{V^P_{T_n}(bi)}$$



(a) Policy P2 (b) Policy P3 (c) Policy P

Figure 15: Relative differences of the policy values of two consecutive points among the belief grid.

Figure 15 shows the relative differences of accumulated cost from the second period to T_{end} of two consecutive points among the belief grid (3003 points). One can notice that these differences are insignificant for all three policies. Therefore, the solution errors of three policies can be negligible.

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